Solution to MATH 116 INTERMEDIATE CALCULUS III
FINAL EXAM

Problem 1. Suppose that a function $f = f(x, y, z)$ has continuous second order partial derivatives and $f(x, y, z) = g(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$. Evaluate $f_{xx} + f_{yy} + f_{zz}$ at the point $(2, -2, 1)$ given that $g'(3) = 6$ and $g''(3) = 1$.

Solution. We have

$$f_x = g'(r)\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \quad f_y = g'(r)\frac{y}{\sqrt{x^2 + y^2 + z^2}}, \quad f_z = g'(r)\frac{z}{\sqrt{x^2 + y^2 + z^2}},$$

$$f_{xx} = g''(r)\frac{x^2}{x^2 + y^2 + z^2} + g'(r)\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}\right) = g''(r)\frac{x^2}{x^2 + y^2 + z^2} + g'(r)\frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}}$$

Similarly,

$$f_{yy} = g''(r)\frac{y^2}{x^2 + y^2 + z^2} + g'(r)\frac{x^2 + z^2}{(x^2 + y^2 + z^2)^{3/2}}, f_{zz} = g''(r)\frac{z^2}{x^2 + y^2 + z^2} + g'(r)\frac{x^2 + y^2}{(x^2 + y^2 + z^2)^{3/2}}$$

Therefore,

$$f_{xx} + f_{yy} + f_{zz}|_{(2,-2,1)} = g''(3) + g'(3)\frac{2}{3} = 5$$

Problem 2. Show that the curve $\vec{r}(t) = \sqrt{t}\ \vec{i} + \sqrt{t}\ \vec{j} - \dfrac{t+3}{4}\vec{k}$ is normal to the surface $x^2 + y^2 - z = 3$ at the intersection point.

Solution. The curve and the surface intersect at point $P(1, 1, -1)$. (To see it we have to solve the equation $t + t + \frac{t+3}{4} = 3$.) The normal vector to the tangent plane to the paraboloid $f(x, y) = x^2 + y^2 - z - 3$ at point P is $\vec{n_1} = \nabla f(1, 1, -1) = 2\vec{i} + 2\vec{j} - \vec{k}$. The tangent line to the curve $\vec{r}(t) = \sqrt{t}\ \vec{i} + \sqrt{t}\ \vec{j} - \dfrac{t+3}{4}\vec{k}$ at point P has the directional vector $\vec{n_2} = \dfrac{dr}{dt}(1) = \dfrac{1}{2}\vec{i} + \dfrac{1}{2}\vec{j} - \dfrac{1}{4}\vec{k}$. Since $\vec{n_1}||\vec{n_2}$ then the given curve is orthogonal to the paraboloid.

Problem 3. Let $F = (3x^2y + z^2)\,\vec{i} + (x^3 - 2yz)\,\vec{j} + (2xz - y^2)\,\vec{k}$,
$\vec{r_1}(t) = t\,\vec{i} + t^4\,\vec{j} + t^3\,\vec{k}$, $0 \le t \le 1$ and $\vec{r_2}(t) = \sin 4t\ \vec{i} + \tan 2t\ \vec{j} + \dfrac{8t}{\pi}\ \vec{k}$, $0 \le t \le \pi/8$.
Let C_n be the graph of the function $\vec{r_n}(t)$, $n = 1$ or $n = 2$.
Show that $\displaystyle\oint_{C_1} F \cdot dr = \oint_{C_2} F \cdot dr$.

Solution. Note that curves C_1 and C_2 have the same initial point $A(0, 0, 0)$ and the same terminal point $B(1, 1, 1)$. Since $F = \nabla f$, where $f = x^3y + z^2x - y^2z$, then F is a conservative vector field. Therefore, both integrals $\displaystyle\oint_{C_1} F \cdot dr$ and $\displaystyle\oint_{C_2} F \cdot dr$ are equal to the same value $f(B) - f(A) = 1$.

Problem 4. Let C be the boundary of the rectangle having vertices at the points $(0,0,0)$, $(0,3,3)$, $(1,3,3)$, $(1,0,0)$ oriented in the clockwise direction when viewed from above. Find the circulation $\oint_C F \cdot dr$ of the vector field $F = x^2\vec{i} + 4xy^3\vec{j} + y^2x\vec{k}$ around the curve C.

Solution. By Stoke's Theorem,

$$\oint_C F \cdot dr = \int\int_S (\nabla \times F) \circ \vec{n} d\sigma,$$

where

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 4xy^3 & y^2x \end{vmatrix} = 2yx\vec{i} - y^2\vec{j} + 4y^3\vec{k}$$

The rectangle belongs to the plane $f(x,y,z) = y - z = 0$. Its projection to the xy-plane is a rectangle $0 \leq x \leq 1, 0 \leq y \leq 3$. Therefore,

$$\vec{n} = \frac{1}{\sqrt{2}}\vec{j} - \frac{1}{\sqrt{2}}\vec{k}, \qquad (\nabla \times F) \circ \vec{n} = -\frac{y^2}{\sqrt{2}} - \frac{4y^3}{\sqrt{2}}, \qquad d\sigma = \frac{|\nabla f|}{|\nabla f \circ \vec{k}|} dA = \sqrt{2} dA$$

Thus,

$$\oint_C F \cdot dr = \int_0^1 \int_0^3 (-y^2 - 4y^3) dy dx = -90.$$

Problem 5. Let D be the region given by $x^2 + y^2 + z^2 \leq 4a^2$ and $x^2 + y^2 \geq a^2$. Let S be the surface of the solid D. Evaluate the flux $\int\int_S F \cdot \vec{n}\ d\sigma$ of the vector field $F = (x + yz)\vec{i} + (y - xz)\vec{j} + (z - e^x \sin y)\vec{k}$ across the surface S.

Solution. By Divergence Theorem, $\int\int_S F \cdot \vec{n}\ d\sigma = \int\int\int_D div(F) dV$. We have,

$$div(F) = \frac{\partial}{\partial x}(x - yz) + \frac{\partial}{\partial y}(y - xz) + \frac{\partial}{\partial z}(z - e^x \sin y) = 3.$$

Therefore,

$$\int\int_S F \cdot \vec{n}\ d\sigma = 3 \int\int\int_d dV = 3 \int_0^{2\pi} \int_{\pi/6}^{5\pi/6} \int_{a/\sin\phi}^{2a} \rho^2 \sin\phi d\rho d\phi d\theta = 12\pi a^3 \sqrt{3}.$$