BİLKENT UNIVERSITY
Mathematics Department

Math 116 Intermediate Calculus III
Summer School 2006-2007

FINAL EXAM

6:00 pm - 8:00 pm (120 minutes) *July 18, 2007*

Surname : ....

Name : ....

Id. No. : ....

Section : ....

IMPORTANT

- This exam consists of 5 questions of equal weight.
- Each question is on a separate sheet. Please read the questions carefully and write your answers under the corresponding question. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get the full credit.
- Calculators are not allowed.

Please do not write anything below this line.

Q1	Q2	Q3	Q4	Q5	Total
20 pts.	20 pts.	20 pts.	20 pts.	20 pts.	100 pts.

Question 1 (20 points).

(a) Evaluate the flux integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$, where $\mathbf{F}(x,y,z) = x^2\mathbf{i} + z\mathbf{k}$, S is the parametric surface $x = \sin u\cos v$, $y = \sin u \sin v$ and $z = \cos u$ for $0 \le u \le \pi$ and $0 \le v \le 2\pi$, and $\mathbf{n}$ is the outward unit normal vector to S.

Solution. The parametric surface S is nothing but the unit sphere given by $x^2+y^2+z^2 = 1$. If D is the solid given by $x^2+y^2+z^2 \le 1$, then the divergence theorem implies that

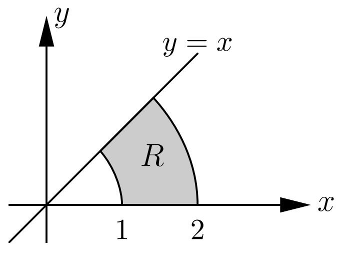
$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}d\sigma &= \iiint_D \text{div}\mathbf{F}\,dV\\ &= \iiint_D (2x+1)\,dV\\ &= \iiint_D 2x\,dV + \iiint_D 1\,dV\end{aligned}$$

Notice that $\iiint_D 2x\,dV = 2\int_0^{\pi}\left(\int_0^{2\pi}\left[\int_0^1 \rho\sin\phi\cos\theta\rho^2\sin\phi\,d\,\rho\right]d\,\theta\right)d\,\phi = 0$

and $\iiint_D 1\,dV = \text{Volume}(D) = \frac{4}{3}\pi$. Thus $\iint_S \mathbf{F}\cdot\mathbf{n}d\sigma = \frac{4}{3}\pi$

(b) Let R be the region in the first quadrant bounded by $x^2+y^2 = 1$, $x^2+y^2 = 4$, $y = 0$ and $y = x$. Evaluate $\oint_C (-y^3 + \mathrm{e}^{x^2})\,dx + (x^3 + \sin(y^2))\,dy$, where C is the boundary of R traced in the counterclockwise direction.

Solution. By Green's Theorem



$$\begin{aligned} I &= \iint_R \left(\frac{\partial}{\partial x}(x^3+\sin(y^2)) - \frac{\partial}{\partial y}(-y^3+e^{x^2})\right)\,dA\\ &= \iint_R (3x^2+3y^2)\,dx\,dy = 3\int_0^{\pi/4}\int_1^2 r^2\,r\,dr\,d\theta\\ &= 3\int_0^{\pi/4} \left.\frac{r^4}{4}\right|_1^2 d\theta = 3\int_0^{\pi/4}\frac{15}{4}\,d\theta = \frac{45\pi}{16}\end{aligned}$$

Question 2 (20 points). Let $\mathbf{F}$ be the vector field defined by $\mathbf{F}(x, y, z) = (2xyz + 3y^2)\mathbf{i} + (x^2z + 6xy - 2z^3)\mathbf{j} + (x^2y - 6yz^2)\mathbf{k}$.

(a) Show that $\mathbf{F}$ is conservative.

Solution. Denote $M = 2xy^2 + 3y^2$, $N = x^z + 6xy - 2z^3$ and $P = x^2y - 6yz^2$.

$$\frac{\partial M}{\partial y} = 2xz + 6y \text{ and } \frac{\partial N}{\partial x} = 2xz + 6y, \text{ i.e, } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial z} = 2xy \text{ and } \frac{\partial P}{\partial x} = 2xy, \text{ i.e, } \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}$$

$$\frac{\partial N}{\partial z} = x^2 - 6z^2 \text{ and } \frac{\partial P}{\partial p} = x^2 - 6z^2, \text{ i.e, } \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}$$

Therefore, $\mathbf{F}$ is conservative.

(b) Evaluate $\int_C (2xyz + 3y^2)\, dx + (x^2z + 6xy - 2z^3)\, dy + (x^2y - 6yz^2)\, dz$, where C is the curve $\mathbf{r} = t^2\mathbf{i} + (t + \sqrt{t})\mathbf{j} + e^t \cos\left(\frac{\pi}{2}t\right)\mathbf{k}$, $0 \leq t \leq 1$.

Solution. Notice that
$\int_C (2xyz + 3y^2)\, dx + (x^2z + 6xy - 2z^3)\, dy + (x^2y - 6yz^2)\, dz = \int_C \mathbf{F} \cdot d\,\mathbf{r}$ where $\mathbf{F}$ is the conservative vector field given in **(a)**. So, $\int_C \mathbf{F} \cdot d\,\mathbf{r}$ is independent of path and indeed, $\int_C \mathbf{F} \cdot d\,\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0))$ for some potential function f (i.e, $\mathbf{F} = \nabla f$). Now, we will a potential function f to $\mathbf{F}$.

$\frac{\partial f}{\partial x} = 2xyz + 3y^2$, i.e, $f(x, y, z) = x^2yz + 3xy^2 + g(z, y)$.
$\frac{\partial f}{\partial y} = x^2z + 6xy - 2z^3$, so $\frac{\partial g}{\partial y} = -2z^3$. Thus, $g(x, y) = -2z^3y + h(z)$
$\frac{\partial f}{\partial z} = x^2y - 6yz^2$, so $h'(z) = 0$, i.e, $h(z) = c$ for some constant c.

Thus, $f(x, y, z) = x^2yz + 3xy^2 - 2z^3y + c$.

$$\int_C \mathbf{F} \cdot d\,\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(1, 2, 0) - f(0, 0, 1) = 12.$$

Question 3 (20 points). Evaluate the flux integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{F}(x, y, z) = xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}$, S is the part of the sphere $x^2 + y^2 + z^2 = 25$ with $z \geq 3$, and $\mathbf{n}$ is the unit normal vector to S away from the origin.

Solution. S is the surface parametrized by $\mathbf{r}(u, v) = (u, v, \sqrt{25 - u^2 - v^2})$ on the domain $D = \{(u, v) \in \mathbb{R}^2 |\, u^2 + v^2 \leq 16\}$. Then

$$d\,\sigma = \|\mathbf{r}_u \times \mathbf{r}_v\| d\,u\; d\,v = \frac{5}{\sqrt{25 - u^2 - v^2}} d\,u\; d\,v$$

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|} = \frac{u}{5}\mathbf{i} + \frac{u}{5}\mathbf{j} + \frac{\sqrt{25 - u^2 - v^2}}{5}\mathbf{k}$$

is the unit normal vector to S away from the origin.

Thus $\quad \mathbf{F} \cdot \mathbf{n}\, d\,\sigma = (u^2 + v^2 + 1)\, du\, dv \quad$ and hence,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_D (u^2 + v^2 + 1) d\,u\; d\,v = \int_0^{2\pi} \Big(\int_0^4 (r^2 + 1) r\, d\,r \Big) d\,\theta = 144\pi.$$

Remark:
Note that the surface is **not** closed and hence the divergence theorem does **not** apply! If you wish to use the divergence theorem you have to close the spherical cap by the disc at $z = 3$.

Question 4 (20 points). Let $F(u, v)$ be a function of u and v. Assume the equation $F(x + y + z, x^2 + y^3 + z^2) = 0$ defines z as a function of x and y, say $z = g(x, y)$. Given that

$$g(1,2) = -1, \quad \left.\frac{\partial F}{\partial u}\right|_{(u,v)=(2,10)} = 3, \quad \left.\frac{\partial F}{\partial v}\right|_{(u,v)=(2,10)} = 12,$$

find the directional derivative of $g(x, y)$ at the point $(1, 2)$ in the direction of the vector $\mathbf{w} = 3\mathbf{i} + 4\mathbf{j}$.

Solution. Denote $u = x + y + z$ and $v = x^2 + y^3 + z^2$. So, $F(x + y + z, x^2 + y^3 + z^2) = F(u, v) = 0$. By differentiating both sides with respect to x, we have

$$\begin{aligned} \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} &= 0 \\ \frac{\partial F}{\partial u}\left(1 + \frac{\partial z}{\partial x}\right) + \frac{\partial F}{\partial v}\left(2x + 2z\frac{\partial z}{\partial x}\right) &= 0 \\ 3\left(1 + \frac{\partial z}{\partial x}\right) + 12\left(2 - 2\frac{\partial z}{\partial x}\right) &= 0 \end{aligned}$$

Thus, $\frac{\partial z}{\partial x} = \frac{9}{7}$ at $(1, 2)$.

By differentiating both sides of $F(u, v) = 0$ with respect to y, we have

$$\begin{aligned} \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} &= 0 \\ \frac{\partial F}{\partial u}\left(1 + \frac{\partial z}{\partial y}\right) + \frac{\partial F}{\partial v}\left(3y^2 + 2z\frac{\partial z}{\partial y}\right) &= 0 \\ 3\left(1 + \frac{\partial z}{\partial y}\right) + 12\left(12 - 2\frac{\partial z}{\partial y}\right) &= 0 \end{aligned}$$

Thus, $\frac{\partial z}{\partial y} = 7$ at $(1, 2)$ and hence $\nabla g|_{(1,2)} = \frac{9}{7}\mathbf{i} + 7\mathbf{j}$

$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$. Therefore,

$(D_{\mathbf{u}}g)|_{(1,2)} = \nabla g|_{(1,2)} \cdot \mathbf{u} = \frac{223}{35}$.

Question 5 (20 points). Find all the local maxima, local minima and saddle points of the function

$$f(x, y) = x^3 + y^3 + 3x^2 - 3y^2.$$

Solution. $f_x = 3x^2 + 6x = 0$ i.e, $x = 0$ or $x = -2$. Also, $f_y = 3y^2 - 6y =$ i.e, $y = 0$ or $y = 2$. So the critical points are: $P_1(0, 0)$ $P_2(0, 2)$ $P_3(-2, 0)$ and $P_4(-2, 2)$. On the other hand, $f_{xx} = 6x + 6$, $f_{xy} = 0$ and $f_{yy} = 6y - 6$.

Consider $\Delta = f_{xx}f_{yy} - (f_{xy})^2$ at the critical points.

$\Delta(P_1) = -36 < 0$, so $P_1(0, 0)$ is a saddle point.

$\Delta(P_2) = 36 < 0$ and $f_{xx}(P_2) = 6$, so there is a local minimum value of f at $P_2(0, 2)$.

$\Delta(P_3) = 36 < 0$ and $f_{xx}(P_3) = -6$, so there is a local maximum value of f at $P_3(-2, 0)$.

$\Delta(P_4) = -36 < 0$, so $P_4(-2, 2)$ is a saddle point.