BİLKENT UNIVERSITY
Mathematics Department

Math 116 Intermediate Calculus III
Summer School 2006-2007

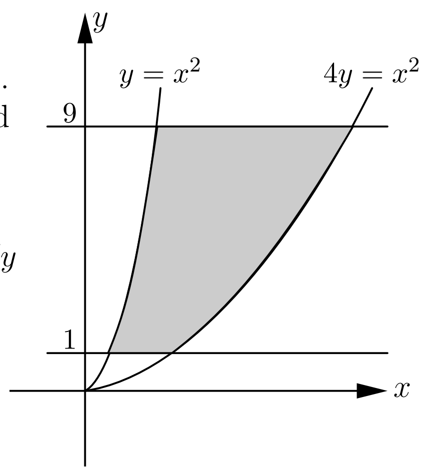
SECOND MIDTERM EXAM

10:00 am - 12:00 pm (120 minutes) *June 30, 2007*

Question 1. (10+10=20 points)

(a) Find the area of the region in the first quadrant bounded by the parabolas $y = x^2$, $4y = x^2$ and the lines $y = 1$, $y = 9$.

Solution. First we draw the picture of the region. In the first quadrant, $y = x^2$ becomes $x = \sqrt{y}$ and $4y = x^2$ becomes $x = 2\sqrt{y}$. So



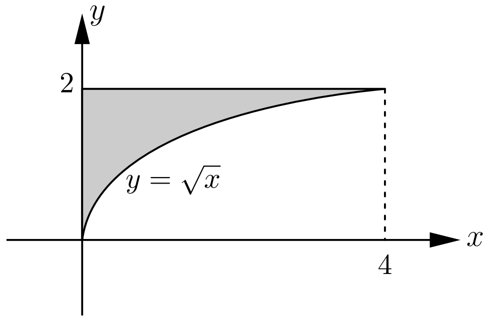
$$A = \int_1^9 \int_{\sqrt{y}}^{2\sqrt{y}} dx\, dy = \int_1^9 x\Big|_{\sqrt{y}}^{2\sqrt{y}} dy = \int_1^9 \sqrt{y}\, dy$$

$$= \frac{2}{3} y^{3/2}\Big|_1^9 = \frac{2}{3}(27-1) = \frac{52}{3}$$

(b) Sketch the region of integration for $I = \int_0^4 \int_{\sqrt{x}}^2 \sqrt{x}\, \mathrm{e}^{y^4}\, \mathrm{d}y\, \mathrm{d}x$ and then evaluate the double integral.

Solution. We reverse the order of integration. $y = \sqrt{x}$ becomes $x = y^2$. So

$$
\begin{aligned}
I &= \int_0^2 \int_0^{y^2} \sqrt{x}\, e^{y^4}\, dx\, dy \\
&= \int_0^2 \frac{2}{3} x^{3/2} e^{y^4} \Big|_0^{y^2} dy = \frac{2}{3} \int_0^2 y^3\, e^{y^4}\, dy \\
&= \frac{2}{3} \cdot \frac{1}{4}\, e^{y^4} \Big|_0^2 = \frac{e^{16} - 1}{6}.
\end{aligned}
$$

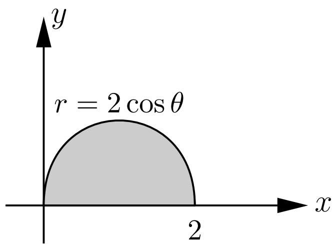


Question 2 (10+10=20 points)

(a) Evaluate $I = \displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} y\sqrt{x^2+y^2}\, \mathrm{d}y\, \mathrm{d}x.$

Solution. We use polar coordinates.
$y = \sqrt{2x - x^2} \Rightarrow x^2 + y^2 = 2x \Rightarrow r = 2\cos\theta$. So

$$
\begin{aligned}
I &= \int_0^{\pi/2} \int_0^{2\cos\theta} r\sin\theta \cdot r \cdot r\, dr\, d\theta \\
&= \int_0^{\pi/2} \sin\theta \cdot \frac{r^4}{4} \Big|_0^{2\cos\theta} d\theta = \int_0^{\pi/2} \sin\theta \cdot \frac{16}{4} \cdot \cos^4\theta\, d\theta \\
&= -4 \cdot \frac{\cos^5\theta}{5} \Big|_0^{\pi/2} = \frac{4}{5}.
\end{aligned}
$$

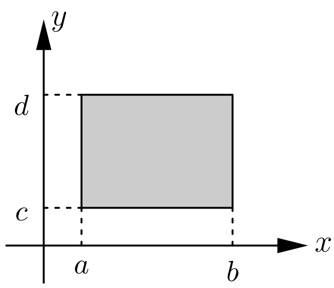


(b) Let R be the rectangular region $a \le x \le b, \quad c \le y \le d$. Prove that for any twice continuously differentiable function $f(x, y)$ one has:

$$
\iint_R \frac{\partial^2 f(x,y)}{\partial x \partial y}\, \mathrm{d}A = f(b,d) - f(a,d) - f(b,c) + f(a,c).
$$

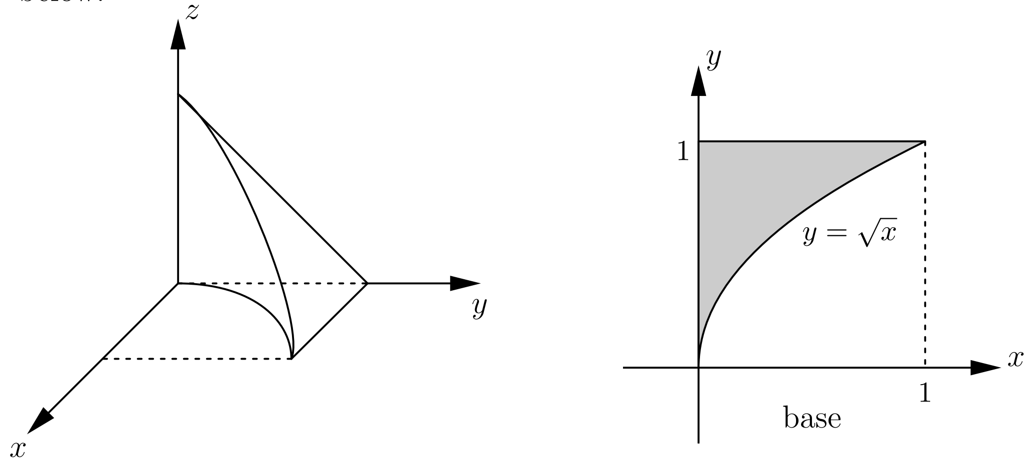
Solution.

$$
\begin{aligned}
\iint_R \frac{\partial^2 f(x,y)}{\partial x \partial y}\, dA &= \int_c^d \int_a^b \frac{\partial^2 f(x,y)}{\partial x \partial y}\, dx\, dy \\
&= \int_c^d \frac{\partial f(x,y)}{\partial y} \Big|_{x=a}^{x=b} dy \\
&= \int_c^d \left(\frac{\partial f(b,y)}{\partial y} - \frac{\partial f(a,y)}{\partial y} \right) dy \\
&= f(b,y)|_c^d - f(a,y)|_c^d. \\
&= f(b,d) - f(b,c) - f(a,d) + f(a,c)
\end{aligned}
$$



Question 3 (20 points) Evaluate $I = \iiint_D x \, dV$ where D is the solid in the first octant bounded by $z = 0$, $z = 1 - y$, $y = \sqrt{x}$ and $x = 0$.

Solution. The pictures of the solid and its base in the xy-plane are shown below.



$$\begin{aligned} I &= \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} x \, dz \, dy \, dx = \int_0^1 \int_{\sqrt{x}}^1 x(1-y) \, dy \, dx \\ &= \int_0^1 x \left(y - \frac{y^2}{2} \right) \Big|_{\sqrt{x}}^1 dx = \int_0^1 x \left(1 - \frac{1}{2} - \sqrt{x} + \frac{x}{2} \right) \, dx \\ &= \int_0^1 \left(\frac{x}{2} - x^{3/2} + \frac{x^2}{2} \right) \, dx = \frac{x^2}{4} - \frac{2}{5} x^{5/2} + \frac{x^3}{6} \Big|_0^1 = \frac{1}{4} - \frac{2}{5} + \frac{1}{6} = \frac{1}{60} \end{aligned}$$

Question 4 (8+8+6=20 points) Let $I = \int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^{\sqrt{x^2+y^2}} x \, dz \, dx \, dy$.

(a) Write I in cylindrical coordinates.
(b) Write I in spherical coordinates.
(c) Evaluate I by using either **(a)** or **(b)**.

Solution. The solid lies in the first octant,it is bounded by the cylinder $x^2 + y^2 = 4$, the yz-plane and the xz-plane on the side, by the cone $z = \sqrt{x^2 + y^2}$ from the top, and the xy-plane from the bottom. So

(a) $I = \int_0^{\pi/2} \int_0^2 \int_0^r r \cos\theta \; r \, dz \, dr \, d\theta.$

(b) The cone $z = \sqrt{x^2 + y^2}$ becomes $\rho = \frac{\pi}{4}$. The cylinder $x^2 + y^2 = 4$

becomes $\rho = 2/\sin\phi$. So

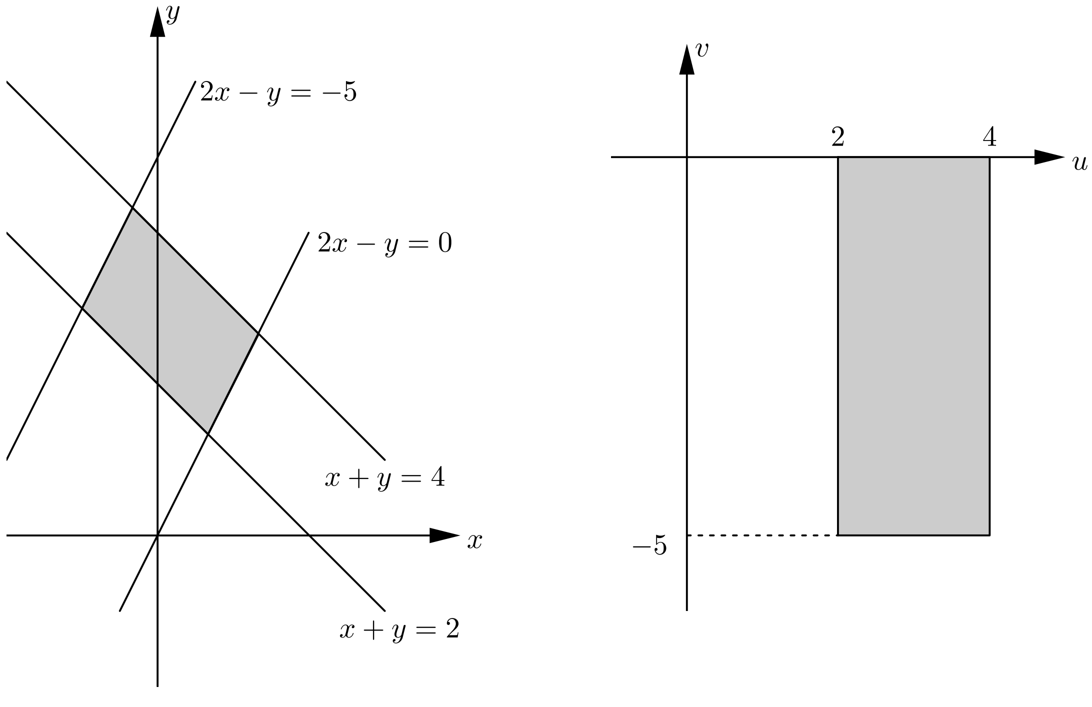
$$I = \int_0^{\pi/2} \int_{\pi/4}^{\pi/2} \int_0^{2/\sin\phi} \rho\sin\phi\ \cos\theta\ \rho^2\ \sin\phi\ d\rho\ d\phi\ d\theta.$$

(c) We use **(a)**.

$$\begin{aligned} I &= \int_0^{\pi/2} \int_0^2 r^2\cos\theta\ z\Big|_0^r\ dr\ d\theta = \int_0^{\pi/2} \int_0^2 r^3\cos\theta\ dr\ d\theta \\ &= \int_0^{\pi/2} \frac{r^4}{4}\Big|_0^2 \cos\theta\ d\theta = 4\int_0^{\pi/2} \cos\theta\ d\theta = 4\sin\theta|_0^{\pi/2} = 4. \end{aligned}$$

Question 5 (20 points) Evaluate $I = \iint_R \frac{\mathrm{e}^{2x-y}}{x+y}\,\mathrm{d}A$, where R is the region bounded by the lines $y = 2x$, $y = 2x+5$, $y = 2-x$ and $y = 4-x$.

Solution. We set $u = x+y$ and $v = 2x-y$. Then $\frac{\partial(u,v)}{\partial(x,y)} = -3$, so $\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{3}$ Now we draw the original region in the xy-plane and the transformed region in the uv-plane.



Then

$$\begin{aligned} I &= \int_2^4 \int_{-5}^0 \frac{e^v}{u}\left|-\frac{1}{3}\right|\ dv\ du = \frac{1}{3}\left(e^v|_{-5}^0\right)\ \left(\ln u|_2^4\right) \\ &= \frac{1}{3}(e^0 - e^{-5})\ (\ln 4 - \ln 2) = \frac{(\ln 2)\ (e^5 - 1)}{3\ e^5}. \end{aligned}$$