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MATH 240, ORDINARY DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 4

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Homework problems from the 8_{nd}, and 9_{nd} Edition of Boyce & DiPrima SECTION 2.4

1. Without solving the D.E. determine an interval in which the solution of the following I.V.P. $y' + (\tan t)y = \sin t, \quad y(\pi) = 0$, is certain to exist.
Solution: The given D.E. is already in normal form and $p(t) = \tan t$. $p(t)$ is discontinuous at $(2n+1)\pi/2$ Since $\pi/2 < x < 3\pi/2$ the initial value problem has a unique solution on the interval $(pi/2,\ 3\pi/2)$.

2. Without solving the D.E. determine an interval in which the solution of the following I.V.P. $(4-t^2)y' + 2ty = 3t^2, \quad y(1) = -3$, is certain to exist.
Solution: Since $p(t) = 2t/(4-t^2)$ and $q(t) = 3t^2/(4-t^2)$. Since both $p(t)$ and $q(t)$ are discontinuous at $x = \pm 2$, the I.V.P. has a unique solution on the interval $(-2, 2)$.

3. For

$$y' = \frac{\ln|ty|}{1 - t^2 + y^2}$$

state where in the $ty-$plane the hypotheses of the Existence and Uniqueness Theorem are satisfied.
Solution: Note that

$$f(t,y) = \frac{\ln|ty|}{1 - t^2 + y^2}$$

is discontinuous along the coordinate axis and on the hyperbola $t^2 - y^2 = 1$. Moreover, $\partial f/\partial y$ has the same points of the discontinuity.

4. For $y' = (t^2 + y^2)^{3/2}$ state where in the $ty-$plane the hypotheses of the Existence and Uniqueness Theorem are satisfied.
Solution: $f(t,y) = (t^2+y^2)^{3/2}$ and $\partial f/\partial y = 3y(t^2+y^2)^{1/2}$ are both continuous everywhere.

5. For

$$\frac{dy}{dt} = \frac{(\cot t)y}{1+y}$$

state where in the $ty-$plane the hypotheses of the Existence and Uniqueness Theorem are satisfied.
Solution: $f(t,y) = y\cot t/(1+y)$ and $\partial f/\partial y = \cot t/(1+y)^2$ are both discontinuous along the lines $t = \pm n\pi$ and $y = -1$.

6. Let

$$y' = \frac{t^2}{y(1+t^3)}, \quad y(0) = y_0.$$

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

Solve the I.V.P. and determine how the interval in which the solution exists depends on the initial value y_0.
Solution: Both $f(t,y) = t^2/[y(1+t^3)]$ and $\partial f/\partial y$ are discontinuous along the straight lines $t=-1$ and $y=0$. The given D.E. is of separable type, if we separate the variables, integrate both sides and using the I.C. $y(0)=y_0$, we obtain

$$y(t) = \left[\frac{2}{3}\ln|1+t^3| + y_0^2\right]^{1/2}.$$

The solution exists if

$$\frac{2}{3}\ln|1+t^3| + y_0^2 \geq 0.$$

i.e. $y_0^2 \geq -\frac{2}{3}\ln|1+t^3|$. For all y_0, (Note that y_0=0 yields a valid solution, even though the E&U theorem does not guarantee one) solution exits as long as $|1+t^3| \geq \exp(-3y_0^2/2)$. From above we must have $t > -1$. Hence the inequality may be written as $t^3 \geq \exp(-3y_0^2/2) - 1$. Then the solution exists for $[\exp(-3y_0^2/2) - 1]^{1/3} < t < \infty$.

7. Solve the following equation: $y' = \epsilon y - \sigma y^3, \quad \varepsilon > 0,\ \delta > 0$. This equation occurs in the study of the stability of fluid flow.
Solution: Bernoulli D.E. with $n=3$. So if we let $v = y^{-2}$, then $y' = -v'(y^2/2)$. Substituting into the D.E. yields the following linear equation for v:

$$v' + 2\varepsilon v = 2\delta.$$

which can be solved by using the integrating factor, and its solution is given as

$$v(t) = 2\sigma t e^{-2\varepsilon t} + c e^{-2\varepsilon t}$$

If we let $y = \pm v^{-1/2}$ we can find the solution $y(t)$ of the given D.E.

8. Solve the following equation: $dy/dt = (\Gamma \cos t + T)y - y^3$,where Γ, and T are constants. This equation also occurs in the study of the stability of fluid flow.
Solution: Bernoulli D.E. with $n=3$. So if we let $v = y^{-2}$, then we obtain the following linear D.E. for v

$$v' + 2(\Gamma\cos t + T)v = 2$$

The solution of the above equation is,

$$v(t) = 2\exp(2\Gamma\sin t - 2Tt)\int_0^t \exp(-2\Gamma\sin x + 2Tx)dx + c\exp(-2\Gamma\sin t + 2Tt).$$

We can find the solution $y(t)$ of the given D.E. by letting $y = \pm v^{-1/2}$.

9. Discontinuous Coefficients. Sometimes the coefficient functions $p(t)$ and $q(t)$ in the linear diff. equations $(y' + p(t)y = q(t))$ have jump discontinuities. If t_0 is the point of discontinuity, then it is necessary to solve the equation separately for $t < t_0$ and $t > t_0$. Afterward, the two solutions are matched so that y is continues at t_0; this is accomplished by a proper choice of the arbitrary constants.
Solve the I.V.P.

$$y' + 2y = g(t), \quad y0) = 0.$$

$g(t) = 1$ when $0 \leq t \leq 1$, and $g(t) = 0$ when $t > 1$.
Solution: For $0 \leq t \leq 1$: Solution of the D.E. $y_1' + 2y_1 = 1$, is $y_1(t) = \frac{1}{2} + c_1 e^{-2t}$. Integration constant

c_1 can be determined by using the I.C. $y(0) = 0$. So the solution of the I.V.P. is $y_1(t) = \frac{1}{2}(1 - e^{-2t})$. So,

$$y(1^-) = y_1(1) = \frac{1}{2}(1 - e^{-2})$$

For $1 < x < \infty$: D.E. is $y_2' + 2y_2 = 0$ and its solution is $y_2 = c_2 e^{-2t}$. Integration constant c_2 can be determined by the continuity of y at $x = 1$. $y(1^+) = y_2(1) = c_2 e^{-2}$. Equating these limits $y(1^-) = y(1^+)$ gives $c_2 = \frac{1}{2}e^{-2} - 1$. Hence the solution of the I.V.P. is

$$y(t) = \frac{1}{2}(1 - e^{-2t}), \quad \text{for } 0 \le t \le 1, \qquad \text{and} \quad y(t) = \frac{1}{2}(e^2 - 1)e^{-2t}, \quad \text{for } t > 1$$

Note the discontinuity in the derivative is the same as the discontinuity in $q(t)$. i.e.

$$y'(t) = e^{-2t}, \quad \text{for } 0 < t < 1, \qquad \text{and} \quad y'(t) = -(e^2 - 1)e^{-2t}, \quad \text{for } t > 1$$

$$y'(1^-) - y'(1^+) = 1$$