

**BILKENT UNIVERSITY**  
**Department of Mathematics**

**MATH 240, ORDINARY DIFFERENTIAL EQUATIONS,**  
**Solution of Homework set<sup>1</sup> # 3**

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**Homework problems from the 8<sup>nd</sup>, and 9<sup>nd</sup> Edition of Boyce & DiPrima**  
**SECTION 2.3**

1. A tank contains 100 gallons of water and 50 oz of salt. Water containing a salt concentration of  $\frac{1}{4}(1 + \frac{1}{2} \sin t)$  oz/gal flows into the tank at a rate of 2 gal/min, and the mixture in the tank flows out at the same rate.

a. Find the amount of salt in the tank at any time.

b. The long time behavior of the solution is an oscillation about a certain constant level. What is this level? What is the amplitude of the oscillation?

**Solution** Let  $Q(t)$  be the amount of salt in the tank at any time  $t$ . Salt enters the tank at a rate of  $2 \left[ \frac{1}{4} \left( 1 + \frac{1}{2} \sin t \right) \right] = \frac{1}{2} + \frac{1}{4} \sin t$ , oz/min and the concentration  $Q(t)/100$ , oz/gal leaves the tank at rate  $2 \cdot Q(t)/100$ , oz/min. Hence the D.E. governing the amount of salt at any time is

$$\frac{dQ}{dt} = \frac{1}{2} + \frac{1}{4} \sin t - \frac{Q}{50}$$

and the I.C.  $Q(0) = Q_0 = 50\text{oz}$ . If we solve the linear D.E. for  $Q$  by using the integrating factor, and using the I.C. we obtain

$$Q(t) = 25 + \frac{1}{2501} [12.5 \sin t - 625 \cos t + 63150e^{-t/50}]$$

The amount of salt approaches a steady state, which is an oscillation of amplitude  $1/4$  about a level of 25 oz.

2. Suppose that a tank containing a certain liquid has an outlet near the bottom. Let  $h(t)$  be the height of the liquid surface above the outlet at time  $t$ . Torricelli's principal states that the outflow velocity  $v$  at the outlet is equal to the velocity of a particle falling freely (with no drag) from the height  $h$ .

a. Show that  $v = \sqrt{2gh}$ , where  $g$  is the gravitational acceleration.

b. By equating the rate of outflow to the rate of change of liquid in the tank, show that  $h(t)$  satisfies the equation

$$A(h) \frac{dh}{dt} = -\alpha a \sqrt{2gh},$$

where  $A(h)$  is the area of the cross section of the tank at height  $h$  and  $a$  is the area of the outlet. The constant  $\alpha$  is a contraction coefficient that accounts for the observed fact that the cross section of the outflow stream is smaller than  $a$ . The value of  $\alpha$  for water is about 0.6.

c. Consider the water tank is the form of right circular cylinder that is 3 m high above the outlet.

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<sup>1</sup>I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

The radius of the tank is 1 m and the radius of the circular outlet is 0.1 m. If the tank initially full of water, determine how long it takes to drain the tank down to the level of the outlet.

**Solution:**

a. Use the "Principal of Conservation of Energy". The speed  $v$  of a particle falling from a height  $h$  is given by  $\frac{1}{2}mv^2 = mgh$

b. The volumetric outflow rate is (*outflowcross-sectionarea*)  $\times$  (*outflowvelocity*):  $\alpha a\sqrt{2gh}$ . At any time, the volume of the water in the tank is

$$V(h) = \int_0^h A(u)du.$$

The rate of change of the volume w.r.t. time  $t$  is

$$\frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt} = A(h) \frac{dh}{dt}$$

Since the volume is decreasing

$$\frac{dV}{dt} = -\alpha a\sqrt{2gh}.$$

c. With  $A(h) = \pi$ ,  $a = 0.01\pi$ ,  $\alpha = 0.06$ , the D.E. for the water level  $h$  is

$$\pi \frac{dh}{dt} = -0.006\pi\sqrt{2gh}.$$

The solution of the D.E. with the I.C.  $h(0) = 3$  and  $g = 9.8$  is given by

$$h(t) = 0.0001764t^2 - 0.046t + 3$$

Then  $h(t) = 0$  for  $t \approx 130.4$  sec.

3. A rocket sled having an initial speed of 150 mi/hr is slowed by a channel of water. Assume that, during the braking process, the acceleration  $a$  is given by  $a(v) = -\mu v^2$ , where  $v$  is the velocity and  $\mu$  is a constant.

a. Use the relation  $dv/dt = (dx/dt)(dv/dx) = v(dv/dx)$  to write the equation of motion in terms of  $v$  and  $x$ .

b. If it requires the distance of 2000 ft to slow the sled to 15 mi/hr, determine the value of  $\mu$ .

c. Find the time  $\tau$  required to slow the sled to 15 mi/hr.

**Solution:**

a. Setting  $-\mu v^2 = v \frac{dv}{dx}$ , yields

$$\frac{dv}{dx} = -\mu v$$

b. The speed  $v$  of the sled satisfies

$$\ln\left(\frac{v}{v_0}\right) = -\mu x$$

Noting that the unit conversion factors cancel, solution of

$$\ln\left(\frac{15}{150}\right) = -2000\mu$$

result in  $\mu = \ln(10)/2000 \text{ ft}^{-1} \approx 6.0788 \text{ mi}^{-1}$ .

c. Solution of  $v' - \mu v^2$  is

$$\frac{1}{v} - \frac{1}{v_0} = \mu t$$

Since  $1 \text{ mile/hr} \approx 1.467 \text{ ft/sec}$ , the elapsed time is

$$t = \frac{1}{(1.467)(0.00115)} \left[ \frac{1}{5} - \frac{1}{150} \right] \approx 35.56 \text{ sec.}$$

4. A mass of  $0.25 \text{ kg}$  is dropped from rest in a medium offering a resistance of  $0.2|v|$ , where  $v$  is the velocity of mass in  $m/sec$ .

a. If the mass is dropped from the height of  $30 \text{ m}$ , find its velocity when it hits the ground.

b. If the mass is to attain a velocity of no more than  $10 \text{ m/sec}$ , find the maximum height from which it can be dropped.

c. Suppose that the resistive force is  $k|v|$ , where  $v$  is measured in  $m/sec$ , and  $k$  is a constant. If the mass is dropped from a height of  $30$ , and must hit the ground with a velocity of no more than  $10 \text{ m/sec}$ , determine the required resistance coefficient  $k$ .

**Solution:**

a. In terms of the displacement, the D.E. is

$$mv \frac{dv}{dx} = -kv + mg$$

By using the chain rule  $\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}$ . Then the D.E. is of separable type and its solution is

$$x(v) = -\frac{mv}{k} - \frac{m^2g}{k^2} \ln \left| \frac{mg - kv}{mg} \right|.$$

Inverse exists, since both  $x$  and  $v$  are monotone increasing. In term of the given parameters,  $x(v) = -1.25v - 15.31 \ln |0.0816v - 1|$ .

b.  $x(0) = 13.45 \text{ m}$ . The required value is  $k = 0.24$ .

c. In part (a), set  $v = 10 \text{ m/sec}$  and  $x = 10 \text{ m}$ .