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MATH 240, ORDINARY DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 1

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Some problems from the 8nd, and 9nd Edition of Boyce & DiPrima SECTION 2.1

1. Solve the following I.V.P. : $y' + 2y = te^{-2t}, \quad y(1) = 0.$
Solution: Given equation is linear, first order, and non-homogenous D.E. which can be solved by using the integrating factor μ. Since $p(t) = 2$, then the integrating factor

$$\mu(t) = \exp\left[\int p(t)dt\right] = e^{2t}.$$

After multiplying the D.E. with $\mu(t)$, the D.E. can be written as

$$\frac{d}{dt}\left(e^{2t}y\right) = t$$

Integrating both sides of the above equation, one obtains

$$y(t) = \frac{1}{2}t^2e^{-2t} + ce^{-2t}, \qquad c = \text{constant}$$

I.C. $t_0 = 1$ and $y_0 = 0$ gives

$$\frac{1}{2}e^{-2} + ce^{-2} = 0,$$

and hence $c = -1/2$. Therefore, the solution of the I.V.P. is

$$y(t) = \frac{t^2 - 1}{2}e^{-2t}.$$

2. Solve the following I.V.P. : $y' + \frac{2}{t}y = \frac{\cos t}{t^2}, \quad y(\pi) = 0, \quad t > 0.$
Solution: Similar to the previous problem, the integrating factor is

$$\mu(t) = \exp\left[\int p(t)dt\right] = \exp\left[\int 2/t\, dt\right] = t^2.$$

Then the general solution of the D.E. is

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)dt + c\right], \qquad c = \text{constant}$$

where $q(t) = \frac{\cos t}{t^2}$. Hence,

$$y(t) = \frac{\sin t}{t^2} + ct^{-2} \qquad c = \text{constant}.$$

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

I.C. implies that $c = 0$. Therefore, the solution of the I.V.P is

$$y(t) = \frac{\sin t}{t^2}$$

3. Solve the following I.V.P. : $y' - 2y = e^{2t}, \quad y(0) = 2.$
Solution: Integrating factor $\mu(t) = e^{-2t}$, and the general solution of the D.E.

$$y(t) = e^{2t}\left[\int dt + c\right] = e^{2t}(t + c), \qquad c = \text{constant}$$

I.C. gives $c = 2$. So the solutio of the I.V.P. is

$$y(t) = (t + 2)e^{2t}.$$

4. Solve the following I.V.P. : $t^3y' + 4t^2y = e^{-t}, \quad y(-1) = 0.$
Solution: First write the D.E. in NORMAL FORM. i.e. multiply the D.E. with t^{-3}. Then $p(t) = 4t^{-1}$ and $q(t) = t^{-3}e^{-t}$. Integrating factor

$$\mu(t) = \exp\left[\int 4t^{-1}dt\right] = t^4.$$

and the solution

$$y(t) = t^{-4}[-(t + 1)e^{-t} + c], \qquad c = \text{constant}$$

I.C. implies that $c = 0$, therefore the solution of the I.V.P. is

$$y(t) = -(t^{-3} + t^{-4})e^{-t}].$$

5. Consider the I.V.P. : $y' - \frac{3}{2}y = 3t + 2e^t, \quad y(0) = y_0$. Find the value of y_0 that separates the solution that grows positively as $t \to \infty$ from those that grow negatively. HoW does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$.
Solution: Integrating factor $\mu(t) = \exp\left[\int -(3/2)dt\right] = e^{-3t/2}$, and then the general solution of the D.E. is

$$y(t) = -2t - \frac{4}{3} - 4e^t + ce^{3t/2}, \qquad c = \text{constant}$$

I.C. gives $c = y_0 + (16/3)$, therefore

$$y(t) = -2t - \frac{4}{3} - 4e^t + (y_0 + \frac{16}{3})e^{3t/2}.$$

As $t \to \infty$, the term containing $e^{3t/2}$ will grow faster than all the other terms. Its sign will determine the divergence properties. Hence the critical value of y_0 is $y_0 = -16/3$. Then the corresponding solution is

$$y(t) = -2t - \frac{4}{3} - 4e^t$$

will also decrease without bound.

6. Construct a first order linear diff. eq. whose solutions have the required behavior as $t \to \infty$. Then solve the equation and confirm that the solutions do indeed have the specified property.

Solution: Let THE DOMINANT BEHAVIOR of y be given by $f(t)$, and consider the function

$y(t) = y_1(t) + f(t)$ such that $y_1(t) \to 0$, as $t \to \infty$ (i.e. $y(t) \to f(t)$, as $t \to \infty$) . For $\alpha > 0$ and constant, write

$$y' + \alpha y = y_1' + \alpha y_1 + f' + \alpha f.$$

Note that, by the hypothesis y_1 will satisfy

$$y_1' + \alpha y_1 = 0.$$

That is,

$$y_1(t) = ce^{-\alpha t}$$

where c is an integration constant, and $\alpha > 0$. Note that $y_1(t) \to 0$, as $t \to \infty$. Hence,

$$y(t) = ce^{-\alpha t} + f(t)$$

which is a solution of the equation

$$y' + \alpha y = f' + \alpha f,$$

For the simplicity let $\alpha = 1$.

6.a. All solutions are asymptotic to the line $y = 3 - t$ as $t \to \infty$..
Solution: For $f(t) = 3 - t$, then then we have the following equation for y:

$$y' + y = 2 - t$$

Solving the above equation by using the integrating factor μ yields the following general solution

$$y(t) = 3 - t + ce^{-t}, \qquad c = \text{constant},$$

which has the given property. i.e. $y(t) \to 3 - t$, as $t \to \infty$.

6.b. All solutions approach the curve $y = 4 - t^2$ as $t \to \infty$..
Solution: In this case, $f(t) = 4 - t^2$, and consider the equation

$$y' + y = 4 - 2t - t^2$$

Solving the above equations gives(integrating factor $\mu(t) = e^t$)

$$y(t) = 4 - t^2 + ce^{-t}, \qquad c = \text{constant}.$$

7. VARIATION OF PARAMETERS Consider the non-homogenous first order linear D.E. $y' + p(t)y = q(t)$. The corresponding homogenous D.E. can be written by setting $q(t) = 0$, i.e. the corresponding homogenous D.E. is $y' + p(t)y = 0$, and which is a separable type D.E. The solution of the corresponding homogenous D.E. is,

$$y = c \exp\left[-\int p(t)dt\right], \qquad c = \text{integration constant}$$

Lets look at the solution of the non homogenous D.E. of the following form:

$$y = c(t) \exp\left[-\int p(t)dt\right],$$

where c is now a function of t. i.e. we let c vary w.r.t. t. Substituting y in the given D.E. yields the following equation for $c(t)$:

$$c'(t) = q(t) \exp\left[\int p(t)dt\right].$$

By solving the above D.E., one can find $c(t)$, and hence the solution of the given non homogenous D.E.. This method is called variation of Parameters.

7.a. Find the general solution of $ty' + 2y = \sin t, \quad t > 0$ by using the variation of parameters.
Solution: If we write the equation in normal form

$$y' + \frac{2}{t}y = \frac{1}{t}\sin t$$

the the corresponding homogenous equation (which is separable type) is

$$y' + \frac{2}{t}y = 0$$

and its solution $y_h(t)$ is

$$y_h(t) = ct^{-2}, \qquad c = \text{constant}.$$

Now, look at the solution of the given D.E. of the following form

$$y(t) = C(t)t^{-2}$$

If we take the derivative of y and substitute into D.E., we obtain the following D.E. for $C(t)$

$$C'(t) = t\sin t$$

Therefore, $C(t) = \sin t - t\cos t + c$ where c is an integration constant. Hence the solution of the given D.E. is

$$y(t) = \frac{1}{t^2}(\sin t - t\cos t + c).$$