BİLKENT UNIVERSITY
Department of Mathematics

MATH 240, DIFFERENTIAL EQUATIONS, Solution of Homework set[1] # 9

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1) Since t^2 is continuous for $0 \le t \le A$ for any positive A and since $t^2 \le e^{at}$ for any $a > 0$ and t for sufficiently large, it follows from the theorem $\mathcal{L}\{t^2\}$ exists for $s > 0$.

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-st}t^2dt = \lim_{A\to\infty}\int_0^A e^{-st}t^2dt = \lim_{A\to\infty}\left\{\left[\frac{-t^2}{s}e^{-st}\right]_0^A + \frac{2}{s}\int_0^A e^{-st}tdt\right\}$$

If we use the integration parts once more, we get

$$\mathcal{L}\{t^2\} = \frac{2}{s^2}\lim_{A\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^A = \frac{2}{s^3}.$$

2) From the definition of $\cosh bt$ we have $\mathcal{L}\{\cosh bt\} = \mathcal{L}\{\frac{1}{2}[e^{(a+b)t}+e^{-(a+b)t}]\}$. By using the linearity of $\mathcal{L}$ we have

$$\mathcal{L}\{\cosh bt\} = \frac{1}{2}\mathcal{L}\{e^{(a+b)t}\} + \frac{1}{2}\mathcal{L}\{e^{-(a+b)t}\}$$

Then we can use the Laplace transform of the exponential function and obtain

$$\mathcal{L}\{\cosh bt\} = \frac{1/2}{s-(a+b)} + \frac{1/2}{s-(a-b)} = \frac{s-a}{(s-a)^2-b^2}, \qquad \text{for} \quad s-a > |b|.$$

3) Similar to previous problem, we use the linearity of the Laplace transform and get

$$\mathcal{L}\{e^{at}\sin bt\} = \mathcal{L}\{\frac{1}{2i}[e^{(a+ib)t} - e^{-(a+ib)t}]\} = \frac{1}{2i}\left[\mathcal{L}\{e^{(a+ib)t}\} - \mathcal{L}\{e^{-(a+ib)t}\}\right].$$

Each of these two terms can be evaluated by using the Laplace transform of the exponential function. We now have to require s to be greater than the real part of the complex numbers $a + ib$ and $a - ib$ in order for the integrals to converge.

$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}, \qquad s > a$$

4) As in the previous problem $\mathcal{L}\{t\sin at\} = (1/2i)\mathcal{L}\{te^{iat}\} - (1/2i)\mathcal{L}\{te^{-iat}\}$. Using the definition of the Laplace transform and integration by parts yield

$$\mathcal{L}\{t\sin at\} = \frac{1}{2i}\left[(s-b)^{-2} - (s+b)^{-2}\right]$$

where $b = ia$ and $s > 0$. Hence

$$\mathcal{L}\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}, \qquad s > 0.$$

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there are any.** If you find any errors and/or misprints, please notify me.

5)

a) Rewrite $F(s)$ as

$$F(s) = 2\frac{2!}{(s-1)^{2+1}}$$

and thus the inverse Laplace transform of $F(s)$ is

$$\mathcal{L}^{-1}\{F(s)\} = 2t^2e^t$$

b) The partial fraction decomposition yields

$$F(s) = \frac{3s}{s^2 - s - 6} = \frac{3s}{(s-3)(s+2)} = \frac{9}{5}\frac{1}{s-3} + \frac{6}{5}\frac{1}{s+2}.$$

The linearity of the inverse Laplace transform we obtain

$$\mathcal{L}^{-1}\{F(s)\} = \frac{9}{5}e^{3t} + \frac{6}{5}e^{-2t}.$$

6)

a) Take the Laplace transform of the D.E. to get

$$s^2Y(s) - sy(0) - y'(0) - [sY(s) - y(0)] - 6Y(s) = 0.$$

Using the I.C. and solving for $Y(s)$ we obtain

$$Y(s) = \frac{s-2}{s^2 - s - 6}$$

We can rewrite $Y(s)$ by using the partial fraction decomposition as follows

$$Y(s) = \frac{s-2}{(s+2)(s-3)} = \frac{4}{5}\frac{1}{s+2} + \frac{1}{5}\frac{1}{s-3}.$$

If we take the inverse Laplace transform of $Y(s)$ we obtain

$$y(t) = \frac{1}{5}(e^{3t} + 4e^{-2t}).$$

b) Taking the Laplace transform, we have

$$s^2Y(s) - sy(0) - y'(0) - 4[sY(s) - y(0)] + 4Y(s) = 0$$

Using the I.C. and solving for $Y(s)$ we obtain

$$Y(s) = \frac{s-3}{s^2 - 4s + 4}$$

Since the denominator is a perfect square, the partial fraction form is

$$Y(s) = \frac{s-3}{s^2 - 4s + 4} = \frac{a}{s-2)^2} + \frac{b}{s-2}$$

Solving a and b yields $a = -1$ and $b = 1$. Thus

$$Y(s) = \frac{s-3}{s^2-4s+4} = \frac{-1}{s-2)^2} + \frac{1}{s-2}$$

Applying inverse Laplace transform gives

$$y(t) = e^{2t} - te^{2t}.$$

c) Note that

$$Y(s) = \frac{2s-4}{s^2-2s-2} = \frac{2s-4}{(s-1)^2-3} = \frac{2(s-1)}{(s-1)^2-3} - \frac{2}{(s-1)^2-3}.$$

By using $\mathcal{L}\{e^{ct}f(t)\} = F(s-c)$ in $\mathcal{L}\{\sinh at\} = a/(s^2-a^2), \quad s > |a|$ and $\mathcal{L}\{\cosh at\} = s/(s^2-a^2), \quad s > |a|$ we obtain

$$y(t) = 2e^t \cosh \sqrt{3}t - \frac{2}{\sqrt{3}} \sinh \sqrt{3}t.$$

d) The Laplace transform of the D.E. is

$$s^2Y(s) - sy(0) - y'(0) + w^2Y(s) = \frac{s}{s^2+4}.$$

Applying the I.C. and solving for $Y(s)$ we obtain

$$Y(s) = \frac{s}{(s^2+4)(s^2+w^2)} + \frac{s}{s^2+w^2}.$$

Decomposing the first term by partial fraction we have

$$Y(s) = \frac{s}{(s^2+4)(s^2+w^2)} - \frac{a}{(s^2+4)(s^2+w^2)} + \frac{s}{s^2+w^2} = \frac{1}{w^2-4}\left[\frac{(w^2-5)s}{s^2+w^2} + \frac{s}{s^2+4}\right].$$

Then applying the inverse Laplace transform, we get

$$y(t) = \frac{1}{w^2-4}\left[(w^2-5)\cos wt + \cos 2t\right].$$

e) Transformed function $Y(s)$ satisfies

$$Y(s) = \frac{1}{(s-1)^2+1} + \frac{1}{(s+1)[(s-1)^2+1]}.$$

Using the partial fraction on the second term we obtain

$$Y(s) = \frac{1}{(s-1)^2+1} + \frac{1}{5}\left[\frac{1}{s+1} - \frac{s-3}{(s-1)^2+1}\right]$$

$$Y(s) = \frac{1}{5}\left[\frac{1}{s+1} - \frac{s-1}{(s-1)^2+1} + \frac{7}{(s-1)^2+1}\right].$$

Hence

$$y(t) = \frac{1}{5}\left(e^{-t} - e^{t}\cos t + 7e^{t}\sin t\right).$$

d) If we apply the Laplace transform on both sides of the D.E. the L.H.S of the D.E. gives $s^2Y(s) - sy(0) - y'(0) + 4Y(s)$. To transform the R.H.S. we must use the definition of the Laplace transform. Since $f(t)$ is piecewise continuous $\mathcal{L}\{f(t)\}$ exists and

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt = \int_0^\pi e^{-st}dt + \lim_{A\to\infty}\int_\pi^A e^{-st}(0)dt,$$

$$\mathcal{L}\{f(t)\} = \frac{1}{s}\,(1 - e^{-\pi s}).$$

Hence, the transformed function $Y(s)$ is given as

$$Y(s) = \frac{s}{s^2+4} + \frac{1-e^{-\pi s}}{s(s^2+4)}.$$

7) Lets write $f(t)$ in terms of the unit step function $u_c(t)$. By completing the square $t^2 - 2t + 2 = (t-1)^2 + 1$, thus we can write

$$f(t) = u_1(t)g(t-1), \qquad \text{where,} \qquad g(t) = t^2 + 1.$$

Hence,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_1(t)g(t-1)\} = e^{-s}\mathcal{L}\{g(t)\} = e^{-s}\left(\frac{2}{s^3} + \frac{1}{s}\right).$$

8) Use the partial fraction

$$F(s) = \frac{1}{3}\,e^{-2s}\left[\frac{1}{s-1} - \frac{1}{s+2}\right]$$

For the simplicity let us define the functions

$$G(s) = \frac{1}{s-1} \qquad \text{and} \qquad H(s) = \frac{1}{s+2}.$$

Then

$$F(s) = \frac{1}{3}\,\left[G(s)e^{-2s} - H(s)e^{-2s}\right].$$

Using the fact that $\mathcal{L}\{e^{at}\} = (s-a)^{-1}$, we have

$$F(s) = \frac{1}{3}\,\left[e^{-2s}\mathcal{L}\{e^{t}\} - e^{-2s}\mathcal{L}\{e^{-2t}\}\right].$$

Thus

$$F(s) = \frac{1}{3}\,\left[\mathcal{L}\{u_2(t)e^{(t-2)}\} - \mathcal{L}\{u_2(t)e^{-2(t-2)}\}\right].$$

Using the linearity of the Laplace transform we have

$$\mathcal{L}\{f(t)\} = \frac{1}{3}\,\mathcal{L}\{u_2(t)[e^{(t-2)} - e^{-2(t-2)}]\}$$

Hence

$$f(t) = \frac{1}{3}\,u_2(t)[e^{(t-2)} - e^{-2(t-2)}]$$

An alternate method is to complete the square in the denominator:

$$F(s) = \frac{e^{-2s}}{(s+1/2)^2 - 9/4}.$$

This gives

$$f(t) = \frac{2}{3}\, u_2(t) e^{-(t-2)/2} \sinh \frac{3}{2}(t-2).$$

which is the seme as that found above.

9)

a) $f(t)$ can be written as $f(t) = 1 - u_{\pi/2}(t)$ and the Laplace transform of the D.E. is

$$(s^2+1)Y(s) - sy(0) - y'(0) = \frac{1}{s}\,(1 - e^{-\pi s/2})$$

Using the I.C. and solving for $Y(s)$, we obtain

$$Y(s) = \frac{1}{s^2+1} + \frac{1}{s(s^2+1)} - e^{-\pi s/2}\frac{1}{s(s^2+1)}.$$

Using partial fraction on the second and third term we find

$$Y(s) = \frac{1}{s} + \frac{1}{s^2+1} - \frac{s}{s^2+1} - \frac{e^{-\pi s/2}}{s} + \frac{se^{-\pi s/2}}{s^2+1}.$$

The inverse Laplace transform of the first three terms can be obtained directly. To find inverse transform of the last two terms we have

$$\mathcal{L}^{-1}\{e^{-s/2}/s\} = u_{\pi/2}(t)g(t-\pi/2)$$

where $g(t) = \mathcal{L}^{-1}\{1/s\} = 1$, and

$$\mathcal{L}^{-1}\{se^{-s/2}/(s^2+1)\} = u_{\pi/2}(t)h(t-\pi/2)$$

where $h(t) = \mathcal{L}^{-1}\{s/(s^2+1)\} = \cos t$. Hence

$$y(t) = 1 + \sin t - \cos t + u_{\pi/2}(t)[\cos(t-\pi/2) - 1] = 1 + \sin t - \cos t - u_{\pi/2}(t)[1 - \sin t].$$

b) Since $\mathcal{L}\{u_{2\pi}\sin(t-2\pi)\} = e^{-2\pi s}\mathcal{L}\{\sin t\} = e^{-2\pi s}/(s^2+1)$. Transforming the D.E.. we have

$$(s^2+4)Y(s) - sy(0) - y'(0) = \frac{1}{s^2+1} - e^{-2\pi s}\,\frac{1}{s^2+1}.$$

Using the I.C. and solving for $Y(s)$, we obtain

$$Y(s) = \frac{1 - e^{-2\pi s}}{(s^2+1)(s^2+4)}.$$

We apply partial fraction to write

$$Y(s) = \frac{1}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4} - \frac{e^{-2\pi s}}{s^2+1} + \frac{e^{-2\pi s}}{s^2+4}\right].$$

Applying the inverse Laplace transform yields

$$y(t) = \frac{1}{3}\left\{\sin t - \frac{1}{2}\sin 2t - u_{2\pi}(t)\left[\sin(t-2\pi) - \frac{1}{2}\sin 2(t-2\pi)\right]\right\}.$$

or

$$y(t) = \frac{1}{6}\ \left[(2\sin t - \sin 2t)\,(1 - u_{2\pi}(t))\right].$$

c) Taking the Laplace transform, using the I.C. and solving for $Y(s)$, we have

$$Y(s) = \frac{1 - e^{-s/2}}{s^2(s^2 + s + 5/4)}$$

By using partial fraction and completed the square in the denominator of the last term give

$$Y(s) = (1 - e^{-s/2})\left\{\frac{4/5}{s^2} - \frac{16/25}{s} + \frac{(16/25)s - (4/25)}{(s+1/2)^2 + 1}\right\} = (1 - e^{-s/2})H(s).$$

Since the numerator of the last term of $H(s)$ can be written as

$$\frac{16}{25}\left[\frac{s+1}{2} - \frac{3}{4}\right]$$

we see that

$$\mathcal{L}^{-1}\{H(s)\} = h(t) = \frac{4}{25}\left(5t - 4 + 4e^{-t/2}\cos t - 3e^{-t/2}\sin t\right)$$

which yields the desired solution

$$y(t) = h(t) - u_{\pi/2}(t)h(t - \pi/2).$$

c) Note that $f(t) = \sin t - u_\pi(t)\sin t = \sin t + u_\pi(t)\sin(t-\pi)$. The transformed function $Y(s) = \mathcal{L}\{y(t)\}$ is given as

$$Y(s) = (1 + e^{-\pi s})\frac{1}{(s^2+1)(s^2+s+5/4)}$$

The correct partial fraction of the quotient is

$$\frac{as+b}{s^2+1} + \frac{cs+d}{s^2+s+5/4},$$

where $a + c = 0$, $a + b + d = 0$, $(5/4)a + b + c = 0$ and $(5/4)b + d = 1$ by equating the coefficients. Solving for the constants yields the solution

$$y(t) = h(t) + u_\pi(t)h(t-\pi), \qquad h(t) = \frac{4}{17}\left[-4\cos t + \sin t + 4e^{-t/2}\cos t + e^{-t/2}\sin t\right].$$

**I do hope that you enjoyed O.D.E's.
I wish you good luck with your final exams and happy summer break.
My best wishes to all of you...**