BİLKENT UNIVERSITY
Department of Mathematics

MATH 240, DIFFERENTIAL EQUATIONS, Homework set # 8

U.Muğan December 1, 2008

Some selected problems from *Elementary Differential Equations and Boundary Value Problems*, W.E.Boyce, R.C.DiPrima, Fifth Edition

Section 5.6, SERIES SOLUTIONS NEAR A REGULAR SINGULAR POINT, Part I

1) Show that the following given D.E's has a regular singular point at $x = 0$. Determine the C.E. (INDICIAL EQUATION, I.E.), roots of the I.E. and the recursion relation. Find the series solution ($x > 0$) corresponding to the larger root. If the roots are equal do not differ by an integer find the series solution corresponding to the smaller root also.

a) $x^2y'' + xy' + (x^2 - \frac{1}{9})y = 0$.
b) $xy'' + y' - y = 0$.

2) The Bessel equation of order zero is

$$x^2y'' + xy' + x^2y = 0$$

Show that $x = 0$ is a regular singular point; that the roots of the C.E. are $r_1 = r_2 = 0$; and that one solution for $x > 0$ is

$$J_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Notice that the series converges for all x, not just $x > 0$. In particular, $J_0(x)$ is bounded as $x \to 0$. The function $J_0(x)$ is known as the Bessel function of the first kind of order zero.

3) The Bessel equation of order one is

$$x^2y'' + xy' + (x^2 - 1)y = 0$$

a) Show that $x = 0$ is a regular singular point; that the roots of the I.E. are $r_1 = 1$ and $r_2 = -1$; and that one solution

$$J_1(x) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n+1)!n!}$$

Notice that $J_1(x)$ is bounded as $x \to 0$. The function J_1 is known as the Bessel function of the first kind of order one.
b) Show that it is impossible to determine a second solution of the form

$$x^{-1} + \sum_{n=0}^{\infty} b_n x^n, \qquad x > 0.$$

Section 5.7, SERIES SOLUTIONS NEAR A REGULAR SINGULAR POINT, Part II

4) Find all the regular singular points of the following given D.E's. Determine the I.E. and the exponents of the singularity at each regular singular point.

a) $xy'' + 2xy' + 6e^x y = 0.$
b) $x(x-1)y'' + 6x^2 y' + 3y = 0.$
c) $x^2(1-x)y'' - (1+x)y' + 2xy = 0.$

5) Show that

$$(\ln x)y'' + \frac{1}{2}y' + y = 0$$

has a regular singular point at $x = 1$. Determine the roots of the I.E. at $x = 1$. Determine the first three nonzero terms in the series $\sum_{n=0}^{\infty} a_n(x-1)^{r+n}$ corresponding to the larger root. Take $x - 1 > 0$. What would you expect the radius of convergence of the series to be?