BİLKENT UNIVERSITY
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MATH 240, DIFFERENTIAL EQUATIONS, Solution[1] of Homework set # 8

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1)

a) If we put the D.E. in normal form, then $p(x) = x^{-1}$ and $q(x) = 1 - (1/9x^2)$. Thus $x = 0$ is a singular point. Since, $xp(x) \to 1$ and $x^2q(x) \to -1/9$ as $x \to 0$. It follows that $x = 0$ is a regular point. In determining a series solution of the D.E. it is better to leave the D.E. as it is given. If we substitute $y = \sum_{n=0}^{\infty} a_n x^{r+n}$, then we obtain

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{r+n} + \sum_{n=0}^{\infty}(n+r)a_nx^{r+n} + \left(x^2 - \frac{1}{9}\right)\sum_{n=0}^{\infty} a_nx^{r+n} = 0$$

After the shifting the summation index, we obtain the following

$$\begin{array}{l} \left[r(r-1) + r - \frac{1}{9}\right] a_0x^r + \left[(r+1)r + (r+1) - \frac{1}{9}\right] a_1x^{r+1} \\ + \sum_{n=2}^{\infty}\{[(n+r)(n+r-1) + (n+r) - \frac{1}{9}]a_n + a_{n-2}\}x^{n+r} = 0. \end{array} \tag{1}$$

The I.E. is

$$r^2 - \frac{1}{9} = 0$$

with roots $r_1 = 1/3$ and $r_2 = -1/3$. For either values of r it is necessary to take $a_1 = 0$ in order that the coefficient of x^{r+1} be zero. The recursion relation is

$$\left[(n+r)^2 - \frac{1}{9}\right] a_n = -a_{n-2}.$$

for $r = 1/3$ we have

$$a_n = \frac{-a_{n-2}}{\left(n+\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)^2} = -\frac{a_{n-2}}{\left(n+\frac{2}{3}\right)n}, \quad n = 2, 3, 4, ...$$

Since $a_1 = 0$ from the recursion relation $a_3 = a_5 = a_7 = ... = 0$. For the even coefficients it is convenient to let $n = 2m$, $m = 1, 2, 3, ...$ Then

$$a_{2m} = -\frac{a_{m-2}}{2^2m\left(m+\frac{1}{3}\right)}.$$

The coefficient of x^{2m} for $m = 1, 2, 3, ...$ is

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}m!\left(1+\frac{1}{3}\right)\left(2+\frac{1}{3}\right)...\left(m+\frac{1}{3}\right)}.$$

Thus one solution by setting $a_0 = 1$ is

$$y_1(x) = x^{1/3}\left[1 + \sum_{m=1}^{\infty}\frac{(-1)^m}{m!\left(1+\frac{1}{3}\right)\left(2+\frac{1}{3}\right)...\left(m+\frac{1}{3}\right)}\left(\frac{x}{2}\right)^{2m}\right].$$

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there are any.** If you find any errors and/or misprints, please notify me.

Since $r_2 = -1/3 \neq r_1$ and $r_1 - r_2 = 2/3$ is not an integer. So we can calculate the second L.I. solution corresponding to $r_2 = -1/3$. The recursion relation is

$$a_{n-2} = -n\left(n - \frac{2}{3}\right) a_n$$

which yields the desired solution following the steps just outlined above. Note that $a_1 = 0$, as in the first solution, and thus all the odd coefficients are zero.

$$y_2(x) = x^{-1/3}\left[1 - \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\left(1-\frac{1}{3}\right)\left(2-\frac{1}{3}\right)...\left(m-\frac{1}{3}\right)}\left(\frac{x}{2}\right)^{2m}\right].$$

b) $x = 0$ is the regular singular point of the D.E. If we substitute $y = \sum_{n=0}^{\infty} a_n x^{r+n}$, into D.E. and shifting the index yield the following indical equation and the recursion relation

$$r^2 = 0, \qquad (r+n+1)^2 a_{n+1} - a_n = 0, \quad n = 0, 1, 2, 3, ...$$

So $r = 0$ is a double root. Thus we will obtain only one series of the form $y = \sum_{n=0}^{\infty} a_n x^{r+n}$. For $r = 0$ the recursion recursion relation gives $a_0 = a_1,\ a_2 = a_1/2^2 = a_0/2^2,\ a_3 = a_2/3^2 = a_0/2^2.3^2,\ ...a_n = a_0/(n!)^2$. Thus one solution, by setting $a_0 = 1$ is

$$y = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

2) If we put the D.E. in normal form, then $p(x) = x^{-1}$ and $q(x) = 1$. Thus $x = 0$ is a singular point. Since, $xp(x) \to 1$ and $x^2 q(x) \to 0$ as $x \to 0$. It follows that $x = 0$ is a regular point. In determining a series solution of the D.E. it is better to leave the D.E. as it is given. If we substitute $y = \sum_{n=0}^{\infty} a_n x^{r+n}$, then we obtain

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{r+n} + \sum_{n=0}^{\infty}(n+r)a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2}x^{r+n} = 0$$

$$[r(r-1)+r]a_0 x^r + [(1+r)r + 1 + r]a_1 x^{r+1} + \sum_{n=2}^{\infty}[(n+r)^2 a_n + a_{n-2}]x^{n+r} = 0$$

The I.E. is $r^2 = 0$ so $r = 0$ is a double root. It is necessary to take $a_1 = 0$ in order that the coefficient of x^{r+1} be zero. The recursion relation is

$$n^2 a_n = -a_{n-2}, \qquad n = 2, 3, 4, ...$$

Since $a_1 = 0$ it follows that $a_3 = a_5 = a_7 = ... = 0$. For the even coefficients we let $n = 2m,\quad m = 1, 2, 3, ...$ Then

$$a_{2m} = -\frac{a_{2m-2}}{2^2 m^2}$$

So,

$$a_2 = -\frac{a_0}{2^2 1^2},\ a_4 = \frac{a_0}{2^2 2^2 1^2 2^2}, ..., a_{2m} = (-1)^m \frac{a_0}{2^{2m}(m!)^2}.$$

Thus one solution of the Bessel equation of order zero is

$$J_0(x) = 1 + \sum_{m=1}^{\infty}(-1)^m \frac{x^{2m}}{2^{2m}(m!)^2}$$

by setting $a_0 = 1$. Using the ratio test we can show that the the series converges for all x. Also note that $J_0(x) \to 1$ as $x \to 0$.

3)
a) If we put the D.E. in normal form, then $p(x) = x^{-1}$ and $q(x) = (x^2 - 1)/x^2$. Thus $x = 0$ is a singular point. Since, $xp(x) \to 1$ and $x^2 q(x) \to -1$ as $x \to 0$. It follows that $x = 0$ is a regular point. In determining a series solution of the D.E. it is better to leave the D.E. as it is given. If we substitute $y = \sum_{n=0}^{\infty} a_n x^{r+n}$, shifting the summation indices appropriately and collecting coefficients of common powers of x we obtain

$$[r(r-1) + r - 1]a_0 x^r + [(1+r)r + 1 + r - 1]a_1 x^{r+1} + \sum_{n=2}^{\infty} \{[(n+r)^2 - 1]a_n + a_{n-2}\} x^{r+n} = 0.$$

The I.E. $r^2 - 1 = 0$ so the roots are $r_1 = 1$ and $r_2 = -1$. For either value of r it is necessary to take $a_1 = 0$ in order that the coefficient of x^{r+1} be zero. The recursion relation

$$a_{n-2} = -a_n[(n+r)^2 - 1], \quad n = 2, 3, 4, \ldots.$$

Since $a_1 = 0$ it follows that $a_3 = a_5 = a_7 = \ldots = 0$. For the even coefficients we let $n = 2m$, $m = 1, 2, 3, \ldots$ Then

$$a_{2m} = -\frac{a_{2m-2}}{2^2 m(m+1)}, \quad m = 1, 2, 3, \ldots$$

So,

$$a_2 = -\frac{a_0}{2^2 1.2}, \; a_4 = \frac{a_0}{2^2 2^2 1.2.2.3}, \ldots, a_{2m} = (-1)^m \frac{a_0}{2^{2m} m!(m+1)!}.$$

Thus one solution of the Bessel equation of order one (by setting $a_0 = 1/2$)

$$J_1(x) = \frac{x}{2} \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{2^{2m} m!(m+1)!}$$

Using the ratio test we can show that the the series converges for all x. Also note that $J_1(x) \to 0$ as $x \to 0$.

b) For $r = -1$ the recursion relation is

$$[(n-1)^2 - 1]a_n = -a_{n-2}, \quad n = 2, 3, \ldots$$

Substituting $n = 2$ into recursion relation we obtain

$$[(2-1)^2 - 1]a_2 = -a_0$$

Hence, it is not possible to determine a_2 and consequently not possible to find a series solution of the form $x^{-1} \sum_{n=0}^{\infty} b_n x^n$.

4)
a) If we write the equation in normal form then $p(x) = 2$, $q(x) = 6e^x/x$. Thus $x = 0$ is a singular point of the D.E. Since $xp(x) = 2x$ and $x^2 q(x) = 6xe^x$ are analytic at $x = 0$ so $x = 0$ is a regular singular point of the D.E. Next we have $xp(x) \to 0 = p_0$ and $x^2 q(x) \to 0 = q_0$ as $x \to 0$. Thus the I.E. is $r(r-1) + p_0 r + q_0 = r^2 - r = 0$ which has roots $r_1 = 1$ and $r_2 = 0$.

b) The equation has the form of $P(x)y'' + Q(x)y' + R(x)y = 0$ with $P(x) = x(x-1)$, $Q(x) = 6x^2$

and $R(x) = 3$. Since $P(x)$, $Q(x)$ and $R(x)$ are polynomials with no common factors and $P(x) = 0$ when $x = 0,\ 1$. Hence $x = 0, 1$ are the singular points of the D.E.
$x = 0:$ Since, $p(x) = 6x/(x-1)$,and $q(x) = 3/x(x-1)$, then $xp(x) \to 0 = p_0$ and $x^2q(x) \to 0 = q_0$ as $x \to 0$. Therefore, the I.E. is $r(r-1) = 0$ and the roots are $r_1 = 0$ and $r_2 = 1$.
$x = 1:$ Since, $p(x) = 6x/(x-1)$,and $q(x) = 3/x(x-1)$, then $(x-1)p(x) = 6x \to 6 = p_0$ and $(x-1)^2q(x) = 3(x-1)/x \to 0 = q_0$ as $x \to 0$. Therefore, the I.E. is $r(r-1)+6r = r(r+5) = 0$ and the roots are $r_1 = 0$ and $r_2 = -5$.

c) For this D.E.

$$p(x) = -\frac{1+x}{x^2(1-x)}, \quad q(x) = \frac{2}{x(1-x)}$$

and thus $x = 0$ and $x = -1$ are the singular points. Since $xp(x)$ is not analytic at $x = 0$. Hence, $x = 0$ is an irregular singular point. For $x = -1$,

$$(x-1)p(x) = \frac{(1+x)}{x^2}, \quad (x-1)^2q(x) = \frac{2(1-x)}{x}$$

are both analytic at $x = -1$. Therefore $x = -1$ is a regular singular point and that $p_0 = 2$ and $q_0 = 0$.

5) Write the D.E. in normal form, since we are looking for the solution about $x = 1$, multiply the D.E. with $(x-1)^2$ and get

$$(x-1)^2y'' + (x-1)\left[\frac{(x-1)}{2\ln x}\right]y' + \left[\frac{(x-1)^2}{\ln x}\right]y = 0$$

Since $\ln 1 = 0$, $x = 1$ is a singular point. To show it is a regular singular point of the D.E. we must show that $(x-1)/\ln x$ is analytic at $x = 1$ and $(x-1)^2/\ln x = (x-1)[(x-1)/\ln x]$ is also analytic at $x = 1$. We expand $\ln x$ in a Taylor series about $x = 1$ we find that

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - ...$$

Thus $\frac{(x-1)}{\ln x}$ has the following Taylor series expansion about $x = 1$

$$\frac{(x-1)}{\ln x} = \left[1 - \frac{1}{2}(x-1) + \frac{1}{3}(x-1)^2 - ....\right]^{-1} = 1 + \frac{1}{2}(x-1) + ...$$

and hence it is analytic. We can use the above result to obtain the indical equation at $x = 1$. We have

$$(x-1)^2y'' + (x-1)\left[\frac{1}{2} + \frac{1}{4}(x-1) + ...\right]y' + \left[(x-1) + \frac{1}{2}(x-1)^2 + ...\right]y = 0.$$

Thus $p_0 = 1/2$ and $q_0 = 0$ and the I.E. is

$$r(r-1) + \frac{r}{2} = 0$$

Hence $r_1 = 1/2$ and $r_2 = 0$. In order to find the first three non-zero terms in a series solution corresponding to $r_1 = 1/2$, it is better to keep the D.E. in its original form and substitute the above power series for $\ln x$:

$$[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + ...]y'' + \frac{1}{2}y' + y = 0$$

If we substitute

$$y = a_0(x-1)^{1/2} + a_1(x-1)^{3/2} + a_2(x-1)^{5/2} + ...$$

and collect the coefficients of like powers of $(x-1)$ which are then set equal to zero. Then we obtain

$$\frac{6a_1}{4} + \frac{9a_0}{8} = 0, \quad 5a_2 + \frac{5a_1}{8} - \frac{a_0}{12} = 0$$

These equations yield

$$a_1 = -\frac{3a_0}{4}, \qquad a_2 = \frac{53a_0}{480}$$

with $a_0 = 1$ we obtain the solution

$$y_1(x) = (x-1)^{1/2} - \frac{3}{4}(x-1)^{3/2} + \frac{53}{480}(x-1)^{5/2} + ...$$

Since the radius of convergence of the series for $\ln x$ is 1, we would expect $\rho = 1$.