**BİLKENT UNIVERSITY**
**Department of Mathematics**

# MATH 240, DIFFERENTIAL EQUATIONS, Solution of Homework set [1] # 7

U. Muğan December 1, 2008

---

**1)**
**a)** Use the ratio test:

$$\lim_{n\to\infty}\frac{|(2x+1)^{n+1}/(n+1)^2|}{|(2x+1)^n/n^2|}=\lim_{n\to\infty}\frac{n^2}{(n+1)^2}|2x+1|=|2x+1|.$$

Therefore, the series converges absolutely for $|x+(1/2)|<1/2$. At $x=0$ and $x=-1$ the series also converge absolutely. However, for $|x+(1/2)|>1/2$ the series diverges by the ratio test. The radius of convergence is $\rho=1/2$.

**b)** Use the ratio test:

$$\lim_{n\to\infty}\frac{|(n+1)x^{n+1}/2^{n+1}|}{|nx^n/2^n|}=\lim_{n\to\infty}\frac{n+1}{2n}|x|=\frac{|x|}{2}.$$

Therefore, the series converges absolutely for $|x|<2$. For $x=2$ and $x=-2$ the $n^{th}$ term does not approach zero as $n\to\infty$ so the series diverges. Hence, the radius of convergence is $\rho=2$.

**2)**
**a)** Since $f(x)=\sin x$ and Taylor series expansion of $f(x)$ about the point $x_0$ is

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Hence, $f'(x)=\cos x,\quad f''(x)=-\sin x,\quad f'''(x)=-\cos x,....$ Then $f(0)=0,\quad f'(0)=1,\quad f''(0)=0,\quad f'''(0)=-1,....$ The even terms in the series will vanish and the odd terms will alternate in sign. We obtain

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}.$$

From the ratio test it follows that $\rho=\infty$.

**b)** Now $f(x)=x^2$. Hence $f'(x)=2x,\quad f''(x)=2,\quad f'''(x)=0,...,f^{(n)}(x)=0$ for $n>2$. Then $f(-1)=1,\quad f'(-1)=-2,\quad f''(-1)=2$ and

$$x^2=1-2(x+1)+2(x+1)^2/2!=1-2(x+1)+(x+1)^2$$

Since the series terminates after a finite number of terms, it converges for all $x$. Thus $\rho=\infty$.

**c)** Now $f(x)=\ln x$. Hence $f'(x)=1/x,\quad f''(x)=-1/x^2,\quad f'''(x)=1.2/x^3,......,f^{(n)}(x)=$

---

[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there are any.** If you find any errors and/or misprints, please notify me.

$(-1)^{n+1}(n-1)!/x^n$. Then $f(1) = 0, \quad f'(1) = 1, \quad f''(1) = -1, f'''(1) = 1.2, ........., f^{(n)}(1) = (-1)^{n+1}(n-1)!$. Then

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - ... = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n}.$$

Since the series terminates after a finite number of terms, it converges for all $x$. From the ratio test, the series converges absolutely for $|x-1| < 1$. However, the series diverges for $x = 0$ so $\rho = 1$.

**3)** We shift the index of the summation in the first sum by letting $m = n - 1$, we have

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty}(m+1)a_{m+1}x^m.$$

Substituting this into the given equation and letting $m = n$ again, we obtain,

$$\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty}[(n+1)a_{n+1} + 2a_n]x^n = 0.$$

Hence,

$$a_{n+1} = -\frac{2a_n}{n+1}, \qquad \text{for } \ n = 0, 1, 2, 3, ...$$

Thus $a_1 = -2a_0, \quad a_2 = -2a_1/2, = 2^2 a_0/2, \quad a_3 = -2a_2/3 = -2^3 a_0/2.3 = -2^3 a_0/3!....$ and $a_n = (-1)^n 2^n a_0/n!$. Notice that for $n = 0$ this formula reduces to $a_0$ so we can write

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=}^{\infty}(-1)^n\frac{2^n a_0}{n!}x^n = a_0\sum_{n=0}^{\infty}\frac{(-2x)^n}{n!} = a_0 e^{-2x}.$$

**4)**
**a)**

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n-2}x^n$$

Substituting in the D.E., we obtain

$$\sum_{n=0}^{\infty}(n+2)(n-1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

In order to have the starting point the same in all three summations, we let $n = 0$ in the first and third terms to obtain the following

$$(2.1.a_2 - a_0)x^0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - (n+1)a_n]x^n = 0$$

Thus $a_{n+2} = a_n/(n+2)$ for $n = 1, 2, 3, ....$ However, note that recursion relation is also correct for $n = 0$. The even and odd coefficients can be determined independently. We show how to calculate the odd $a$'s: $a_3 = a_1/3, \quad a_5 = a_3/5 = a_1/(5.3), \quad a_7 = a_5/7 = a_1/(7.5.3), ...,$ Note that

$a_3 = 2a_1/(2.3) = 2a_1/3!$, $a_5 = 2.4a_1/(2.3.4.5) = 2^2.2a_1/5!$, $a_7 = 2.4.6a_1/(2.3.4.5.6.7) = 2^3 3!a_1/7!$. Continuing we have

$$a_{2m+1} = \frac{2^m m! a_1}{(2m+1)!}$$

In the same way we find even $a$'s

$$a_{2m} = \frac{a_0}{2^m m!}$$

Thus we have

$$y = a_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{2^m m!} + a_1 \sum_{m=0}^{\infty} \frac{2^m m! x^{2m+1}}{(2m+1)!}.$$

**b)**

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n, \quad y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n-2} (x-1)^n$$

Substituting in the D.E., and replacing the coefficient $x$ by $1 + (x-1)$ we obtain

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n - \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n - \sum_{n=1}^{\infty} n a_n (x-1)^n - \sum_{n=0}^{\infty} a_n (x-1)^n = 0,$$

where the index has been shifted in the second summation. Letting $n = 0$ in the first, and the fourth sums, we obtain

$$(2.1.a_2 - 1.a_1 - a_0)(x-1)^0 + \sum_{n=1}^{\infty} [(n+2)(n+1) a_{n+2} - (n+1) a_{n+1} - (n+1) a_n](x-1)^n = 0.$$

Thus $(n+2)A_{n+2} - a_{n+1} - a_n = 0$ for $n = 0, 1, 2, 3, ...$ This recursion relation can be used to solve for $a_2$ in terms of $a_0$ and $a_1$, the foe $a_3$ in terms of $a_0$ and $a_1$, etc. In many cases it is easier to first take $a_0 = 0$ and generate the one solution and then take $a_1 = 0$ and generate the second L.I. solution. Thus choosing $a_0 = 0$ we find that

$$a_2 = a_1/2, \quad a_3 = (a_2 + a_1)/3 = a_1/2, \quad a_4 = (a_3 + a_2)/4 = a_1/4, ....$$

This yields the solution

$$y_2 = a_1[(x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \frac{3}{20}(x-1)^5 + ...$$

The second L.I. solution can be obtained by choosing $a_1 = 0$. Then

$$a_2 = a_0/2, \quad a_3 = (a_2 + a_1)/3 = a_0/6, \quad a_4 = (a_3 + a_2)/4 = a_0/6, ....$$

This yields the solution

$$y_1 = a_0[1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \frac{1}{15}(x-1)^5 + ...$$

Note that the general solution is the linear combination of $y_1$ and $y_2$.

**c)**

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting in the D.E., and shifting the index in the series for $y''$ gives

$$y = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$(2.1.a_2 + a_0) + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + a_n]x^n = 0$$

Thus $a_2 = -a_0/2$, and

$$a_{n+2} = \frac{na_{n+1}}{(n+2)} - \frac{a_n}{(n+1)(n+2)}$$

Choosing $a_0 = 0$ gives one solution as

$$y_2(x) = a_1\left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 + ...\right)$$

A L.I. solution is obtained by choosing $a_1 = 0$ :

$$y_1(x) = a_0\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + ...\right).$$

**d)** You will need to rewrite $x+1$ as $3+(x-2)$ in order to multiply $x+1$ times $y'$ as a power series about $x_0 = 2$.
Recursion relation:

$$2(n+2)(n+1)a_{n+2} + 3(n+1)a_{n+1} + (n+3)a_n = 0, \qquad n = 0, 1, 2, 3, ...$$

Solutions:

$$y_1(x) = 1 - \frac{3}{4}(x-2)^2 + \frac{3}{8}(x-2)^3 + \frac{1}{64}(x-2)^4 + ....$$
$$y_2(x) = (x-2) - \frac{3}{4}(x-2)^2 + \frac{1}{24}(x-2)^3 + \frac{9}{64}(x-2)^4 + ....$$

**5)** The D.E. transforms into

$$u''(t) + t^2u'(t) + (t^2 + 2t)u(t) = 0$$

Assuming thet $u(t) = \sum_{n=0}^{\infty} a_n t^n$, we have $u'(t) = \sum_{n=1}^{\infty} na_n t^{n-1}$ and $u''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$. Substituting in the D.E. and shifting the indices yields

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=2}^{\infty}(n-1)a_{n-1}t^n + \sum_{n=2}^{\infty} a_{n-2}t^n + \sum_{n=1}^{\infty} 2a_{n-1}t^n = 0$$

If we take first two terms of first and first term of the last sum, we get

$$2.1.a_2 + (3.2.a_3 + 2a_0)t + \sum_{n=2}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)a_{n-1} + a_{n-2}]t^n = 0.$$

It follows that $a_2 = 0$, $a_3 = -a_0/3$ and

$$a_{n+2} = -\frac{a_{n-1}}{n+2} - \frac{a_{n-2}}{(n+2)(n+1)}, \qquad n = 2, 3, 4, ...$$

We obtain one solution by choosing $a_0 = 0$, thus one solution is

$$u_1(t) = a_0\left[1 - \frac{1}{3}t^3 - \frac{1}{12}t^4 + \frac{1}{18}t^6 + ....\right]$$

So

$$y_1(x) = u_1(x-1) = a_0\left[1 - \frac{1}{3}(x-1)^3 - \frac{1}{12}(x-1)^4 + \frac{1}{18}(x-1)^6 + ....\right]$$

We obtain the second L.I. solution by choosing $a_0 = 0$:

$$u_2(t) = a_1\left[t - \frac{1}{4}t^4 - \frac{1}{20}t^5 + \frac{1}{28}t^7 + ....\right]$$

So

$$y_2(x) = u_2(x-1) = a_1\left[(x-1) - \frac{1}{4}(x-1)^4 - \frac{1}{20}(x-1)^5 + \frac{1}{28}(x-1)^7 + ....\right]$$

The Taylor series for $x^2 - 1$ about $x = 1$ may be obtained by writing $x = (x-1) + 1$ so $x^2 = (x-1)^2 + 2(x-1) + 1$ and $x^2 - 1 = (x-1)^2 + 2(x-1)$. The D.E. now appears as

$$y'' + (x-1)^2 y' + [(x-1)^2 + 2(x-1)]y = 0$$

which is identical to the transformed equation with $t = x - 1$.

**6)**

$$y = a_0 + a_1 x + a_2 x^2 + ... \,, \qquad y^2 = a_0^2 + 2a_0 a_1 x + (2a_0 a_2 + a_1^2)x^2 + .....$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + ...., \qquad \text{and} \qquad (y')^2 = a_1^2 + 4a_1 a_2 x + (6a_1 a_3 + 4a_2^2)x^2 + ....$$

Substituting these series in the D.E. and collecting coefficients of like powers of $x$ yields

$$(a_1^2 + a_0^2 - 1) + (4a_1 a_2 + 2a_0 a_1)x + (6a_1 a_3 + 4a_2^2 + 2a_0 a_2 + a_1^2)x^2 + ... = 0$$

Each coefficients must be zero. The I.C. $y(0) = 0$ requires that $a_0 = 0$, and thus $a_1^2 + a_0^2 - 1 = 0$ gives $a_1^2 = 1$. However, $y'(0) = a_1 > 0$ implies $a_1 = 1$. Then $4a_1 a_2 + 2a_0 a_1 = 0$ implies that $a_2 = 0$; and $6a_1 a_3 + 4a_2^2 + 2a_0 a_2 + a_1^2 = 6a_1 a_3 + a_1^2 = 0$ implies that $a_3 = -1/6$. Thus

$$y = x - \frac{1}{3!}x^3 + ....$$

**7)** The D.E. can be solved for $y''$ to yield $y'' = -xy' - y$. If $y = \phi(x)$ is a solution, then

$$\phi''(x) = -x\phi'(x) - \phi(x)$$

and thus setting $x = 0$ we obtain $\phi''(0) = -0 - 1 = -1$. Differentiating the D.E. and solving for $y'''$ yields $y''' = -xy'' - 2y'$ and setting $y = \phi(x)$ again gives $\phi'''(0) = -0 - 0 = 0$. Similarly, $y^{(4)} = -xy''' - 3y''$ and thus $\phi^{(4)}(0) = -0 - 3(-1) = 3$. The process can be continued to calculate higher derivatives of $\phi(x)$.

**8)** The zeros of $P(x) = x^2 - 2x - 3$ are $x = -1$ and $x = 3$. For $x_0 = 4$, $x_0 = -4$, and $x_0 = 0$ the distance to the nearest zero of $P(x)$ is $1, 3,$ and $1$ respectively. Thus a lower bound for the radius of convergence for the series solutions in powers of $(x-4)$, $(x+4)$, and $x$ is $\rho = 1$, $\rho = 3$ and $\rho = 1$ respectively.

**9)** The Taylor series about $x = 0$ for $\sin x$ is

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - ....$$

Assuming that $y = \sum_{n=0}^{\infty} a_n x^n$ we find

$$y'' + (\sin x)y = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + ... + (x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - ....)(a_0 + a_1 x + a_2 x^2 + ....) = 0$$

$$2a_2 + (6a_3 + a_0)x + (12a_4 + a_1)x^2 + (20a_5 + a_2 - \frac{1}{6}a_0)x^3 + (30a_6 + a_3 - \frac{1}{6}a_1)x^4 + .... = 0$$

Hence, $a_2 = 0,\ a_3 = -a_0/6,\ a_4 = -a_1/12,\ a_5 = a_0/120,\ a_6 = (a_1 + a_0)/180, ....$ We set $a_0 = 1$ and $a_1 = 0$ and obtain

$$y_1(x) = \left(1 - \frac{x^3}{6} + \frac{x^5}{120} + \frac{x^6}{180} + ....\right)$$

Next we set $a_0 = 0$ and $a_1 = 1$ and obtain

$$y_2(x) = \left(x - \frac{x^4}{12} + \frac{x^6}{180} + ....\right)$$

**10)** Substituting $y = \sum_{n=0}^{\infty} a_n x^n$ into D.E. and shifting indices in the summation yield

$$y = \sum_{n=0}^{\infty} [(n+1)a_{n+1} - a_n]x^n = x^2$$

Equating coefficients of both sides then gives,

$$a_1 - a_0 = 0, \quad 3a_3 - a_2 = 1, \quad \text{and} \quad (n+1)a_{n+1} = a_n, \quad \text{for} \quad n = 3, 4, 5, ...$$

Thus $a_1 = a_0,\quad a_2 = a_1/2 = a_0/2,\quad a_3 = 1/3 + a_2/3 = 1/3 + a_0/(2.3),\quad a_4 = 1/(3.4) + a_0/(2.3.4), ....$ and

$$a_n = \frac{a_{n-1}}{n} = \frac{2}{n!} + \frac{a_0}{n!}$$

Hence

$$y(x) = a_0\left(1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!}\right) + 2\left(\frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^n}{n!}\right).$$

Using the power series for $e^x$, the first and second sums can be written in closed form.

$$y(x) = a_0 e^x + 2\left(e^x - 1 - x - \frac{x^2}{2}\right) = ce^x - 2 - 2x - x^2.$$

**11)**

**a)** Since the coefficients of $y$, $y'$ and $y''$ have **no common factors** and since $P(x)$ vanishes only at $x = 0$ we conclude that $x = 0$ is a singular point. Writing the D.E. in normal form, we obtain $p(x) = (1 - x)/x$ and $q(x) = 1$. Then

$$\lim_{x \to 0} xp(x) = \lim_{x \to 0}(1 - x) = 1, \quad \text{and} \quad \lim_{x \to 0} x^2 q(x) = 0.$$

Thus $x = 0$ is a regular singular point.

**b)** Write the D.E. in normal form, then

$$p(x) = \frac{x}{(1-x)(1+x)^2}, \qquad q(x) = \frac{1}{(1-x)^2(1+x)}.$$

Therefore, $x = \pm 1$ are the singular points of the D.E. Since,

$$\lim_{x\to 1}(x-1)p(x), \quad \text{and} \quad \lim_{x\to 1}(x-1)^2 q(x)$$

both exist, so $x = 1$ is a regular singular point. Since $\lim_{x\to -1}(x+1)p(x)$ does not exist, $x = -1$ is an irregular singular point.

**c)** If we write the D.E. in normal form, we se that $p(x) = e^x/x$ and $q(x) = (3\cos x)/x$. Thus $x = 0$ is a singular point of the D.E. Since $xp(x) = e^x$ and $x^2 q(x) = 3x\cos x$ both analytic at $x = 0$, the point $x = 0$ is regular singular point of the D.E.

**d)** If we write the D.E. in normal form, we se that $p(x) = x/\sin x$ and $q(x) = 4/\sin x$. Thus $x = \pm n\pi, \ \ n = 0, 1, 2, 3, ...$ are the singular points of the D.E. Now we should determine whether the singular points are regular or irregular. Note that, if

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n (x-x_0)^n, \qquad |x-x_0| < \rho, \quad \rho > 0$$

Then the followings are true for $|x - x_0| < \rho, \quad \rho > 0$
i) The series can be formally multiplied, and

$$f(x)g(x) = \left[\sum_{n=0}^{\infty} a_n (x-x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n (x-x_0)^n\right] = \sum_{n=0}^{\infty} c_n (x-x_0)^n$$

where $c_n = a_0 b_n + a_1 b_{n-1} + ... + a_n b_0$.
ii) If $g(x_0) \neq 0$, the series can be formally divided

$$\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} d_n (x-x_0)^n$$

In most case the coefficients $d_n$ can be most easily obtained by equating coefficients in the equivalent relation

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = \left[\sum_{n=0}^{\infty} d_n (x-x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n (x-x_0)^n\right] = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} d_k b_{n-k}\right)(x-x_0)^n.$$

Also in the case of the division, the radius of convergence of the resulting power series may be less then $\rho$.
Then for $x_0 = 0$:

$$xp(x) = \frac{x^2}{\sin x} = \frac{x^2}{x - \frac{x}{3!} + ...} = x\left[1 + \frac{x^2}{3!} + ...\right] = x + \frac{x^2}{3!} + ...$$

which converges about $x_0 = 0$ and thus $xp(x)$ is analytic at $x_0 = 0$. $x^2 q(x)$, by similar steps, is also analytic at $x_0 = 0$ and thus $x_0 = 0$ is a regular singular point. For $x_0 = n\pi$, we have

$$(x-n\pi)p(x) = \frac{x(x-n\pi)}{\sin x} = \frac{(x-n\pi)[(x-n\pi)+n\pi]}{\pm(x-n\pi) \mp \frac{(x-n\pi)^3}{3!} \pm ...} = [(x-n\pi)+n\pi]\left[\pm 1 \mp \frac{(x-n\pi)^2}{3!} \pm ....\right]$$

which converges about $x_0 = n\pi$ and thus $(x - n\pi)p(x)$ is analytic at $x_0 = n\pi$. Similarly $(x + n\pi)p(x)$ and $(x \pm n\pi)^2 q(x)$ are analytic and thus $x_0 = \pm n\pi$ are the regular singular points.

**12)** Substitute $y = \sum_{n=0}^{\infty} a_n x^n$ into D.E. yields

$$2\sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} + 3\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0.$$

The last sum becomes $\sum_{n=2}^{\infty} a_{n-2} x^{n-1}$ by replacing $n+1$ by $n-1$, the first term of the middle sum is $3a_1$, and thus we have

$$3a_1 + \sum_{n=2}^{\infty} \{[2n(n-1) + 3n]a_n + a_{n-2}\}x^{n-1} = 0$$

Hence, $a_1 = 0$ and

$$a_n = -\frac{a_{n-2}}{n(2n+1)}$$

which is the desired recursion relation. Thus all even coefficients are found in terms of $a_0$ and all odd coefficients are zero, thereby yielding only one solution of the desired form.

**13)** Since $P(x) = x^2$, $Q(x) = x$ and $R(x) = x^2 - \nu^2$ then

$$f(z) = \frac{1}{P(1/z)}\left[2\frac{P(1/z)}{z} - \frac{Q(1/z)}{z^2}\right] = \frac{1}{z}$$

and

$$g(z) = \frac{R(1/z)}{z^4 P(1/z)} = \frac{1}{z^4} - \frac{\nu^2}{z^2}$$

where $z = 1/x$. Thus the point at infinity is a singular point. Since $zf(z) = 1$ is analytic at $z = 0$, $z^2 g(z) = (1/z^2) - \nu^2$ is not, so the point at infinity is an irregular singular point.

**14)**
**a)** Let $y = x^r$ and obtain the C.E. $F(r) = r(r-1) - 5r + 9 = 0$ or $(r-3)^2 = 0$. Thus the roots are $r = 3$ with multiplicity 2. Then the solution is

$$y(x) = c_1 x^3 + c_2 x^3 \ln|x|, \quad x \neq 0, \qquad c_1, c_2 = \text{constant}.$$

**b)** Plese note that there is a misprint in the D.E., and the coefficient of $y''$ should be $2x^2$. Let Let $y = x^r$ and obtain the C.E. $2r(r-1) + r - 3 = (2r-3)(r+1) = 0$. Thus

$$y(x) = c_1 x^{3/2} + c_2 x^{-1} \quad x \neq 0, \qquad c_1, c_2 = \text{constant}.$$

I.C. gives that

$$c_1 + c_2 = 1, \qquad \frac{3}{2}c_1 - c_2 = 4$$

Thus, $c_1 = 2$ and $c_2 = -1$. Hence

$$y(x) = 2x^{3/2} - x^{-1}, \quad x \neq 0.$$

**c)** Substituting $y = x^r$, we find that $r(r-1) + \alpha r + 5/2 = 0$ or $r^2 + (\alpha - 1)r + 5/2 = 0$. Thus

$$r_1, r_2 = \frac{1}{2}\left[-(\alpha - 1) \pm \sqrt{(\alpha - 1)^2 - 10}\right]$$

In order for solutions to approach zero as $x \to 0$ it is necessary that the real parts of $r_1$ and $r_2$ be positive. Suppose that $\alpha > 1$, then $\sqrt{(\alpha-1)^2-10}$ is either imaginary or real and less than $\alpha - 1$; hence the real parts of $r_1$ and $r_2$ will be negative. Suppose that $\alpha = 1$, then $r_1, r_2 = \pm i\sqrt{10}$ and the solutions are oscillatory. Suppose that $\alpha < 1$, then $\sqrt{(\alpha-1)^2-10}$ is either imaginary or real and less than $|\alpha - 1| = 1 - \alpha$; hence the real parts of $r_1$ and $r_2$ will be positive. Thus if $\alpha < 1$ the solution of the D.E. will approach zero as $x \to 0$.