BİLKENT UNIVERSITY
Department of Mathematics

MATH 240, DIFFERENTIAL EQUATIONS, Solution of Homework set[1] # 5

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1)

a) First we find the solution of the corresponding homogenous D.E. $y'' - 2y' - 3y = 0$, which has the C.E. $r^2 - 2r - 3 = (r-3)(r+1) = 0$. Hence $y_h(x) = c_1 e^{3x} + c_2 e^{-x}$, $c_1, c_2 =$ constant, and we assume $y_p = Ae^{2x}$ for the particular solution. Thus $y' = 2Ae^{2x}$ and $y'' = 4Ae^{2x}$ and substituting into the D.E. yields

$$4Ae^{2x} + 2(2Ae^{2x}) - 3(Ae^{2x}) = 3e^{2x}$$

Thus $-3A = 3$ and $A = -1$, yielding $y(x) = c_1 e^{3x} + c_2 e^{-x} - e^{2x}$.

b) Homogenous solution $y_h(x) = c_1 + c_2 e^{-2x}$, $c_1, c_2 =$ constant. Initially we assume $y_p = A + B_1 \sin 2x + B_2 \cos 2x$. However, since a constant is a homogenous solution of the D.E. we must modify y_p by multiplying the constant A by x and thus the correct form of y_p is $y_p = Ax + B_1 \sin 2x + B_2 \cos 2x$. If we take the derivatives if y_p and substitute into D.E. the coefficients of the similar terms give $A = 3/2$, $B_1 = B_2 = -1/2$.

c) Homogenous solution $y_h(x) = c_1 e^{-x} + c_2 x e^{-x}$, $c_1, c_2 =$ constant. Assume $y_p = Ax^2 e^{-x}$, so that $y_p' = 2Axe^{-x} - Ax^2 e^{-x}$ and $y_p'' = 2Ae^{-x} - 4Axe^{-x} + Ax^2 e^{-x}$. Substituting in the d.E. gives

$$(Ax^2 - 4Ax + 2A)e^{-x} + 2(-Ax^2 + 2Ax)e^{-x} + Ax^2 e^{-x} = 2e^{-x}.$$

Note that all terms on the left involving x^2 add to zero and we left with $2A = 2$ or $A = 1$. Hence

$$y = c_1 e^{-x} + c_2 x e^{-x} + x^2 e^{-x}.$$

d) Assume that $y_p = (Ax+B)\sin 2x + (Cx+D)\cos 2x$ which is appropriate for both terms appearing in $R(x)$. Since none of the term in y_p is the homogenous solution, we do not need to modify y_p.
Ans.: $y = c_1 \cos x + c_2 \sin x - \frac{1}{3} x \cos 2x - \frac{5}{9} \sin 2x$, $c_1, c_2 =$ constant.

e) C.E. of the corresponding homogenous equation is $r^2 + r + 4 = 0$. Hence,

$$y_h = e^{-x/2}[c_1 \cos(\sqrt{15}\ x/2) + c_2 \sin(\sqrt{15}\ x/2)], \quad c_1, c_2 = \text{constant}.$$

By using the given hint, assume $y_p = Ae^x + Be^{-x}$. Since neither e^x and e^{-x} are solutions of the homogenous equation, then there is no need to modify y_p. Differentiating y_p and substituting in the D.E. yield

$$6Ae^x + 4Be^{-x} = e^x - e^{-x}.$$

Hence, $A = 1/6$ and $B = -1/4$. So the general solution is

$$y = e^{-x/2}[c_1 \cos(\sqrt{15}\ x/2) + c_2 \sin(\sqrt{15}\ x/2)] + (1/6)e^x - (1/4)e^{-x}.$$

2)

a) $y_h(x) = c_1 e^{-2x} + c_2 e^x$, $c_1, c_2 =$ constant, so for the particular solution we assume $y_p = Ax + B$.

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there are any.** If you find any errors and/or misprints, please notify me.

Since neither Ax nor B are the solutions of the homogenous equation it is not necessary to modify y_p. Substituting y_p in the D.E. we obtain

$$A - 2(Ax + B) = 2x$$

Then the coefficients of the similar terms give $A = -1$ and $B = -1/2$. The general solution is

$$y_h(x) = c_1 e^{-2x} + c_2 e^x - x - 1/2.$$

I.C. $y(0) = 0$ and $y'(0) = 1$ imply $c_1 + c_2 - 1/2 = 0$ and $-2c_1 + c_2 - 1 = 1$ respectively. Solving c_1 and c_2 from above equations gives $c_1 = -1/2$ and $c_2 = 1$.
b) $y_h(x) = c_1 e^{3x} + c_2 e^{-x}$, $c_1, c_2 =$ constant. The particular solution is $y_p = (A = Bx)e^{2x}$. Substituting y_p in the D.E. and equating the coefficients of the similar terms give $A = -2/3,\ B = -1$. Initial conditions imply that $c_=1$, $c_2 = 2/3$ Therefore the general solution of the given I.V.P is

$$y(x) = e^{3x} + (2/3)e^{-x} - [(2/3) + x]e^{2x}.$$

3)
a) $y_h(x) = c_1 e^{-3x} + c_2$, $c_1, c_2 =$ constant. After inspection of the right hand side of the D.E. $R(x)$, we assume

$$y_p = (A_0x^4 + A_1x^3 + A_2x^2 + A_3x + A_4) + (B_0x^2 + B_1x + B_2)e^{-3x} + C\sin 3x + D\cos 3x.$$

However, since e^{-3x} and a constant are the homogenous solutions, we must multiply the coefficient of e^{-3x} and the polynomial by x. The correct form is

$$y_p = x(A_0x^4 + A_1x^3 + A_2x^2 + A_3x + A_4) + x(B_0x^2 + B_1x + B_2)e^{-3x} + C\sin 3x + D\cos 3x.$$

b) $y_h(x) = e^{-x}[c_1\cos x + c_2\sin x]$, $c_1, c_2 =$ constant. From $R(x)$ we can assume

$$y_p = Ae^{-x} + (B_0x^2 + B_1x + B_2)e^{-x}\cos x + (C_0x^2 + C_1x + C_2)e^{-x}\sin x.$$

Since $e^{-x}\cos x$ and $e^{-x}\sin x$ are the homogenous solutions of the D.E., it is necessary to multiply both of these terms by x. Hence the correct form of the y_p is

$$y_p = Ae^{-x} + x(B_0x^2 + B_1x + B_2)e^{-x}\cos x + x(C_0x^2 + C_1x + C_2)e^{-x}\sin x.$$

c) $y_h(x) = e^{-x}[c_1\cos 2x + c_2\sin 2x]$, $c_1, c_2 =$ constant. From $R(x)$ we can assume

$$y_p = (A_0x + A_1)e^{-x}\cos 2x + (B_0x + B_1)e^{-x}\sin 2x + (C_0x + C_1)e^{-2x}\cos x + (D_0x + D_1)e^{-2x}\sin x.$$

Since $e^{-x}\cos 2x$ and $e^{-x}\sin 2x$ are the homogenous solutions of the D.E., it is necessary to multiply both of these terms by x. Hence the correct form of the y_p is

$$y_p = x(A_0x + A_1)e^{-x}\cos 2x + x(B_0x + B_1)e^{-x}\sin 2x + (C_0x + C_1)e^{-2x}\cos x + (D_0x + D_1)e^{-2x}\sin x.$$

4)
a) First solve the corresponding homogenous D.E. $y''' - y'' - y' + y = 0$. Its C.E. is $r^3 - r^2 - r + 1 = 0$ and the roots are $r = -1,\ 1,\ 1$. Hence,

$$y_h = c_1 e^{-x} + c_2 e^x + c_3 x e^x, \qquad c_1,\ c_2,\ c_3 = \text{constant}$$

Using the superposition principal, we can write a particular solution as the sum of particular solutions corresponding to $y''' - y'' - y' + y = 2e^{-x}$, and $y''' - y'' - y' + y = 3$. Therefore $y_{p_1} = Ae^{-x}$, but e^{-x} is a homogenous solution so we should multiply by x. Thus $y_{p_1} = Axe^{-x}$. For the second equation $y_{p2} = B$ since the constant is not a homogenous solution there is no need to modify y_{p_2}. Therefore a particular solution of the given D.E. is $y_p = y_{p_1} + y_{p2} = Axe^{-x} + B$. The constants $A,\ B$ can be determined by substituting y_p into the D.E. and we obtain $A = 1/2$ and $B = 3$. Therefore the general solution of the given non-homogenous equation is

$$y = c_1e^{-x} + c_2e^x + c_3xe^x + (1/2)xe^{-x} + 3$$

b) The C.E. of the corresponding homogenous equation $y''' + 4y' = 0$ is $r^3 + 4r = 0$ with roots $r = 0,\ \pm 2i$. Hence

$$y_h = c_1 + c_2 \cos 2x + c_3 \sin 2x, \qquad c_1,\ c_2,\ c_3 = \text{constant}$$

A particular solution $y_p = Ax + B$, but since constant is a homogenous solution, we should multiply y_p by x and assume $y_p = x(Ax + B)$. The constants $A,\ B$ can be determined by substituting y_p into the D.E. and we obtain $A = 1/8$ and $B = 0$. Thus the general solution is

$$y_h = c_1 + c_2 \cos 2x + c_3 \sin 2x + (1/8)x^2.$$

Applying the I.C. we find $c_1 = 3/16$, $c_2 = -3/16$ and $c_3 = 0$.

5)

a) The C.E. of the corresponding homogenous equation $y''' - 2y'' + y' = 0$ is $r^3 - 2r^2 + r = 0$ with roots $r = 0,\ 1,\ 1$. Hence

$$y_h = c_1 + c_2e^x + c_3xe^x, \qquad c_1,\ c_2,\ c_3 = \text{constant}$$

A particular solution of $y''' - 2y'' + y' = x^3$ is $y_{p_1} = A_0x^3 + A_1x^2 + A_2x + A_3$ but since constant is a homogenous solution, we should take

$$y_{p_1} = x(A_0x^3 + A_1x^2 + A_2x + A_3).$$

A particular solution of $y''' - 2y'' + y' = 2e^x$ is $y_{p_2} = Be^x$, but since both e^x and xe^x are solutions of the homogenous equation, we should multiply y_{p_2} by x^2 to obtain $y_{p_2} = Bx^2e^x$. Then

$$y_p = y_{p_1} + y_{p_2} = x(A_0x^3 + A_1x^2 + A_2x + A_3) + Bx^2e^x.$$

b) The homogenous solution of the corresponding homogenous equation $y^{(4)} - y''' - y'' + y' = 0$ is

$$y_h = c_1 + c_2e^{-x} + c_3e^x + c_4xe^x, \qquad c_1,\ c_2,\ c_3,\ c_4 = \text{constant}$$

Consider $y^{(4)} - y''' - y'' + y' = x^2 + 4$ and $y^{(4)} - y''' - y'' + y' = x \sin x$ separately. A particular solution of the first equation $y_{p_1} = A_0x^2 + A_1x + A_2$ but this must be multiplied by x since constant is a homogenous solution. Hence $y_{p_1} = x(A_0x^2 + A_1x + A_2)$. For the second equation $y_{p_2} = (B_0x + B_1)\cos x + (C_0x + C_1)\sin x$ which does not need to be modified. By the superposition principal a particular solution of the given D.E. is

$$y_p = y_{p_1} + y_{p_2} = x(A_0x^2 + A_1x + A_2) + (B_0x + B_1)\cos x + (C_0x + C_1)\sin x.$$

c) The homogenous solution of the corresponding homogenous equation $y^{(4)} + 4y'' = 0$ is

$$y_h = c_1 + c_2x + c_3 \sin 2x + c_4 \cos 2x, \qquad c_1,\ c_2,\ c_3,\ c_4 = \text{constant}$$

A particular solutions

$$y_{p_1} = x(A_1 \sin 2x + A_2 \cos 2x), \qquad y_{p_2} = (B_1 x + B_0)e^x, \qquad y_{p3} = C_1 x^2.$$

By the superposition a particular solution of the given D.E. is

$$y_p = y_{p_1} + y_{p_2} + y_{p_3} = x(A_1 \sin 2x + A_2 \cos 2x) + (B_1 x + B_0)e^x + C_1 x^2.$$

6)

a) The homogenous solution of the given D.E. is

$$y_h = c_1 + c_2 e^x + c_3 x e^x, \qquad c_1,\ c_2,\ c_3 = \text{constant}$$

Since $R(x) = x^3 + 2e^x$, x^3 suggests that the root of the C.E. of $g(D)$ is $r = 0$ with multiplicity $k = 4$, and $2e^x$ suggests that the root of the C.E. of $g(D)$ is $r = 1$. Therefore, the C.E. of $g(D)$ is $g(r) = r^3(r-1) = 0$, and hence

$$g(D)R(x) = D^4(D-1)R(x)$$

If we apply $g(D)$ on both sides of the given D.E. $f(D)y = R(x)$, we obtain the following homogenous equation

$$g(D)f(D)y = 0.$$

The C.E. of the above D.E. is $r^4(r-1)(r^3 - 2r^2 + r = 0) = 0$ with roots $r = 0$ with multiplicity $k = 5$ and $r = 1$ with multiplicity $k = 3$. Hence

$$y_h = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + c_6 e^x + c_7 x e^x + c_8 x^2 e^x.$$

b) The homogenous solution of the corresponding homogenous equation is $y_h = c_1 + c_2 e^{-x} + c_3 e^x + c_4 x e^x, \quad c_1,\ c_2,\ c_3,\ c_4 = \text{constant}$. Since $R(x) = x^2 + 4 + x \sin x$, the C.E. of the D.E. $g(D)R(x) = 0$ is $g(r) = r^3(r-i)^2(r+i)^2 = 0$. Therefore $g(D) = D^3(D-i)^2(D+i)^2$, and hence

$$g(D)f(D)y = 0,$$

where $f(D) = D^4 - D^3 - D^2 + D$. The C.E. of the above homogenous equation is $r^3(r-i)^2(r+i)^2(r^4 - r^3 - r^2 + r) = 0$ with roots $r = 0$ with multiplicity $k = 4$, $r = \pm i$ with multiplicity $k = 2$, $r = 1$ and $r = -1$ with multiplicity $k = 2$. Hence

$$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + (c_5 + c_6 x)\sin x + (c_7 + c_8 x)\cos x + c_9 e^x + c_{10} e^{-x} + c_{11} x e^{-x}.$$

c) The homogenous solution of the corresponding homogenous equation is $y_h = c_1 + c_2 x + c_3 \sin 2x + c_4 \cos 2x, \quad c_1,\ c_2,\ c_3,\ c_4 = \text{constant}$. Since $R(x) = \sin 2x + xe^x + 4$, the C.E. of the D.E. $g(D)R(x) = 0$ is $g(r) = (r-2i)(r+2i)(r-1)^2 r = 0$.Therefore $g(D) = (D-2i)(D+2i)(D-1)^2 D$, and hence

$$g(D)f(D)y = 0,$$

where $f(D) = D^4 + 4D^2$. The C.E. of the above homogenous equation is $(r-2i)(r+2i)(r-1)^2 r^3 (r^2 + 4) = 0$ with roots $r = \pm 2i$, $r = 1$ with multiplicity $k = 2$ and $r = 0$ with multiplicity $k = 3$. Hence

$$y = (c_1 + c_2 x)\sin 2x + (c_3 + c_4 x)\cos 2x + (c_5 + c_6 x)e^x + c_7 + c_8 x + c_9 x^2.$$