BİLKENT UNIVERSITY
Department of Mathematics

MATH 240, DIFFERENTIAL EQUATIONS, Homework set # 4

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REPEATED ROOTS; REDUCTION OF ORDER

1) Find the general solution of the following D.E's
a) $y'' - 2y + y = 0.$
b) $25y'' - 20y' + 4y = 0.$
c) $y'' - 2y' + 10y = 0.$

2) Solve the following I.V.P. and describe the behavior of the solution for increasing x.
a) $y'' + 4y' + 4y = 0, \qquad y(-1) = 2, \quad y'(-1) = 1.$
b) $9y'' - 12y' + 4y = 0, \qquad y(0) = 2, \quad y'(0) = -1.$
c) $9y'' + 6y' + 82y = 0, \qquad y(0) = -1, \quad y'(0) = 2.$

3) Use the reduction of order to find the second L.I. solution of the following D.E.
a) $x^2y'' + 2xy' - 2y = 0, \quad x > 0, \qquad y_1(x) = x.$
b) $xy'' - y' + 4x^3y = 0, \quad x > 0, \qquad y_1(x) = \sin x^2.$
c) $(x-1)y'' - xy' + y = 0, \quad x > 1, \qquad y_1(x) = e^x.$

4) If y_1 is a known solution of $y'' + p(x)y' + q(x)y = 0$, show that a second solution y_2 satisfies

$$\left(\frac{y_2}{y_1}\right)' = \frac{W(y_1, y_2)}{y_1^2}$$

where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2. Then use

$$W(y_1, y_2)(x) = W_0 \exp\left[-\int p(x)dx\right], \qquad W_0 = \text{constant}$$

to determine y_2.
Use the above method to find a second L.I. solution of the following D.E's.
a) $x^2y'' + 3xy' + y = 0, \quad x > 0, \qquad y_1(x) = x^{-1}.$
b) $x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad x > 0, \qquad y_1(x) = x^{-1/2}\sin x.$

HIGHER ORDER D.E.

5) Let the linear differential operator $f(D)$ be defined by

$$f(D)y = a_0y^{(n)} + a_1y^{(n-1)} + ... + a_ny$$

where $a_i, \quad i = 1, 2, ..., n$ are real constants.
a) Find $f(D)x^n$
b) Find $f(D)e^{mx}$
c) Determine the solutions of $y^{(4)} - 5y'' + 4y = 0$. Do you the four solutions form a fundamental set of solutions? Why?

6) Let

$$y''' + p_1(x)y'' + p_2(x)y' + p_3(x)y = 0$$

and $y_1,\ y_2,\ y_3$ be the solutions of the equation on an interval I

a) If $W = W(y_1,\ y_2,\ y_3)$, show that

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} \tag{1}$$

b) Substitute for y_1''', y_2''', y_3''' from the differential equation, multiply the first row by p_3, the second row by p_2, and add these to the last row to obtain

$$W' = -p_1(x)W.$$

Then show that

$$W(y_1,\ y_2,\ y_3)(x) = W_0 \exp\left[-\int p_1(x)dx\right], \qquad W_0 = \text{constant}.$$

Therefore, we can generalize this argument to the nth order equation

$$y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + ... + p_n(x)y = 0$$

with solutions $y_1, y_2, ...y_n$. That is

$$W(y_1, ...,\ y_n)(x) = W_0 \exp\left[-\int p_1(x)dx\right], \qquad W_0 = \text{constant}.$$

7) (REDUCTION OF ORDER) Show that if y_1 is a solution of

$$y''' + p_1(x)y'' + p_2(x)y' + p_3(x)y = 0$$

then the substitution $y = y_1(x)v(x)$ leads to following D.E. for v':

$$y_1 v''' + (3y_1' + p_1 y_1)v'' + (3y_1'' + 2p_1 y_1' + p_2 y_1)v' = 0$$

a) Use the reduction of order to solve the following D.E.

$$(2-x)y''' + (2x-3)y'' - xy' + y = 0, \quad x < 2, \qquad y_1(x) = e^x.$$

HIGHER ORDER, HOMOGENOUS, CONSTANT COEFFICIENT D.E.

8) Determine the solution of the following D.E's.

a) $y''' - 3y'' + 3y' - y = 0.$

b) $y^{(6)} + y = 0.$

c) $y^{(6)} - y'' = 0.$

d) $y^{(4)} - 8y' = 0.$

e) $y^{(6)} + 8y = 0.$

9) Find the solution of the following I.V.P.

a) $y''' + y' = 0, \qquad y(0) = 0,\ \ y'(0) = 1,\ \ y''(0) = 2.$

b) $y^{(4)} - 4y''' + 4y'' = 0, \qquad y(1) = -1,\ \ y'(1) = 2,\ \ y''(1) = 0,\ \ y'''(1) = 0.$