BİLKENT UNIVERSITY
Department of Mathematics

MATH 240, DIFFERENTIAL EQUATIONS, Solution of Homework set[1] # 3

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1)
a)

$$W(e^{2x}, e^{-3x/2}) = \begin{vmatrix} e^{2x} & e^{-3x/2} \\ 2e^{2x} & -\frac{3}{2}e^{-3x/2} \end{vmatrix} = -\frac{3}{2}e^{7x/2} - 2e^{x/2} \tag{1}$$

b)

$$W(x, xe^x) = \begin{vmatrix} x & xe^x \\ 1 & x + xe^x \end{vmatrix} = xe^x + x^2e^x - xe^x = x^2e^x \tag{2}$$

c)

$$W(e^x \sin x, e^x \cos x) = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x \sin x + e^x \cos x & e^x \cos x - e^x \sin x \end{vmatrix} = e^{2x} \tag{3}$$

2)
a) If $y'' + p(x)y' + q(x)y = g(x)$, then for the given equation

$$p(x) = -\frac{3x}{x-1}, \qquad q(x) = \frac{4}{x-1}, \qquad g(x) = \frac{\sin x}{x-1}$$

So, only point of discontinuity of the given differential equation (D.E.) is $x = 1$. By the existence and uniqueness theorem the largest interval is $-\infty < x < 1$. Since then initial point is at $x_0 = -2$ which is contained by the interval.
b) $p(x) = \cos x, \quad q(x) = 3\ln|x|, \quad g(x) = 0$. So, the largest interval is $0 < x < \infty$ which contains the initial point $x_0 = 2$.
c) $p(x) = \dfrac{1}{x-2}, \qquad q(x) = \tan x, \qquad g(x) = 0$
So, point of discontinuity of $p(x)$ is $x = 2$ and the points of the discontinuities of $\tan x$ are $x = \pm(2n+1)\frac{\pi}{2}, \quad n =$ integer. Therefore, the largest interval which contains the initial point is $2 < x < 3\pi/2$.
3) If you take the derivatives of $y_1(x) = 1$ and substitute into given D.E. equation is identically satisfied. Similarly, for $y_2(x) = x^{1/2}$. If $y = c_1 + c_2x^{1/2}$ is substituted in the given D.E., we get

$$-\frac{1}{4}c_1c_2x^{-3x/2} = 0$$

which is zero only if $c_1 = 0$ or $c_2 = 0$. Thus, the superposition of two L.I. solutions is not, in general a solution. Note that the D.E. is nonlinear.

4)

$$W(e^{2x}, e^{-3x/2}) = \begin{vmatrix} x & g \\ 1 & g' \end{vmatrix} = xg' - g = x^2e^x \tag{4}$$

[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. Please check all the calculations/answers and notify me if you find any errors and/or misprints.

or

$$g' - \frac{1}{x}g = xe^x$$

We have a linear first order equation in $g(x)$, its integrating factor is $\mu(x) = 1/x$, and thus the solution is $g(x) = xe^x + cx, \qquad c = \text{constant}.$

5)

a) If one substitutes $y_1(x) = \cos 2x$ and its derivatives into given D.E., equation is identically satisfied. Similarly, if one substitutes $y_1(x) = \sin 2x$ and its derivatives into given D.E., equation is identically satisfied. So, y_1 and y_2 are the solutions of the given differential equation. Since,

$$W(\cos 2x, \sin 2x) = 2\cos^2 2x + 2\sin^2 2x = 2 \neq 0$$

So, y_1 and y_2 form the fundamental set of solutions of the given D.E.

b) For $y_1 = x$ we have,

$$x^2(0) - x(x+2)(1) + (x+2)x = 0$$

and for $y_1 = xe^x$

$$x^2(x+2)e^x - x(x+2)(x+1)e^x + (x+2)xe^x = 0$$

So, D.E. is identically satisfied for both y_1 and y_2. Since

$$W(x, xe^x) = x^2e^x \neq 0 \quad \text{for } x > 0$$

So, y_1 and y_2 form the fundamental set of solutions of the given D.E.

6)

a) Assume $y = e^{rx}$ where r =constant and to be determined, which is substituted into D.E. to obtain the characteristic equation (C.E.)

$$r^2 + 5r = 0$$

So, the roots are $r_1 = 0$ and $r_2 = -5$. Thus the general solution is

$$y(x) = c_1e^{0x} + c_2e^{-5x} = c_1 + c_2e^{-5x}, \qquad c_1, c_2 = \text{constant}.$$

b) The C.E. is $r^2 - 9r + 9 = 0$. So the roots of the C.E. are

$$r_{1,2} = \frac{1}{2}(9 \pm 3\sqrt{5})$$

Hence the general solution is

$$y(x) = c_1e^{(9+3\sqrt{5})x/2} + c_2e^{(9-3\sqrt{5})x/2}, \qquad c_1, c_2 = \text{constant}.$$

c) The C.E. is $2r^2 - 3r + 1 = 0$. So the roots of the C.E. are

$$r_{1,2} = 1, \; \frac{1}{2}$$

Hence the general solution is

$$y(x) = c_1e^{x/2} + c_2e^x, \qquad c_1, c_2 = \text{constant}.$$

7)
a) The C.E. is $r^2 + 4r + 3 = 0$. So the roots of the C.E. are

$$r_1 = -1, \qquad r_2 = -3$$

Hence the general solution is

$$y(x) = c_1 e^{-x} + c_2 e^{-3x}, \qquad c_1, c_2 = \text{constant}.$$

If we take the derivative of $y(x)$

$$y'(x) = -c_1 e^{-x} - 3c_2 e^{-3x}$$

Substituting $x = 0$ into $y(x)$ and $y'(x)$ gives the following equations for c_1 and c_2

$$c_1 + c_2 = 2, \qquad - c_1 - 3c_2 = -1$$

Therefore, $c_1 = 5/2, \quad c_2 = -1/2$. Hence, the solution of the I.V.P. is

$$y(x) = \frac{5}{2} e^{-x} - \frac{1}{2} e^{-3x}.$$

b) The C.E. is $r^2 + 8r - 9 = 0$. So the roots of the C.E. are

$$r_1 = 1, \qquad r_2 = -9$$

Hence the general solution is

$$y(x) = c_1 e^{x} + c_2 e^{-9x}, \qquad c_1, c_2 = \text{constant}.$$

If we take the derivative of $y(x)$

$$y'(x) = c_1 e^{-x} - 9c_2 e^{-9x}$$

Substituting $x = 1$ into $y(x)$ and $y'(x)$ gives the following equations for c_1 and c_2

$$c_1 e + c_2 e^{-9} = 1, \qquad - c_1 e - 9c_2 e^{-9} = 0$$

Therefore, $c_1 = (9/10)e^{-1}, \quad c_2 = (1/10)e^{9}$. Hence, the solution of the I.V.P. is

$$y(x) = \frac{1}{10}\left(9e^{-1}e^{x} + e^{9}e^{-9x}\right).$$

8)
The general solution of the given D.E. is (you can find the general solution as follows: substitute $y = e^{rx}$ int D.E. and get the C.E. and then find the roots of the C.E.)

$$y(x) = y(x) = c_1 e^{-x} + c_2 e^{2x}, \qquad c_1, c_2 = \text{constant}.$$

By using the initial conditions, we obtain

$$c_1 + c_2 = \alpha, \qquad - c_1 + 2c_2 = 2$$

By adding these two equations for c_1 and c_2 we find,

$$3c_2 = \alpha + 2$$

If $y(x)$ approaches to zero as $x \to \infty$, c_2 must be zero. Thus $\alpha = -2$.

9)

a) Look for the solution $y = e^{rx}$, r =constant. Substitute y into D.E. and get the C.E. $r^2 - 2r + 2 = 0$ which has the roots $r_1 = 1 + i$ and $r_2 = 1 - i$. Thus the real part of r is $\lambda = 1$ and the imaginary part $\mu = 1$. So the general solution is

$$y = c_1 e^x \cos x + c_2 e^x \sin x \qquad c_1, c_2 = \text{constant}.$$

b) In this case the C.E. is $9r^2 + 9r - 4 = 0$. The roots are $r_1 = 1/3$ and $r_2 = -4/3$ and then the general solution is

$$y = c_1 e^{x/3} + c_2 e^{-4x/3} \qquad c_1, c_2 = \text{constant}.$$

c) C.E. is $4r^2 + 9 = 0$. The roots are $r_1 = 3i/2$ and $r_2 = -3i/2$ and then the general solution is

$$y = c_1 \cos(3x/2) + c_2 \sin(3x/2) \qquad c_1, c_2 = \text{constant}.$$

10)

a) C.E. is $r^2 + 4r + 5 = 0$ which has roots $r_1 = -2 \pm i$. Thus,

$$y = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x \qquad c_1, c_2 = \text{constant}$$

and

$$y' = (-2c_1 + c_2) e^{-2x} \cos x + (-c_1 - 2c_2) e^{-2x} \sin x$$

so, that $y(0) = c_1 = 1$ and $y'(0) = -2c_1 + c_2 = 0$ or $c_2 = 2$. Hence,

$$y = e^{-2x}(\cos x + 2 \sin x).$$

Since the real part of the roots λ is -2, the amplitude is exponentially decaying. Hence, $y \to 0$ as $x \to \infty$.

b) C.E. is $r^2 + 2r + 2 = 0$ which has roots $r_1 = -1 \pm i$. Thus,

$$y = e^{-x}(c_1 \cos x + c_2 \sin x) \qquad c_1, c_2 = \text{constant}$$

Since the I.C. are given at $x_0 = \pi/4$ we assume

$$y = e^{-(x-\pi/4)}(c_1 \cos x + c_2 \sin x) \qquad c_1, c_2 = \text{constant}$$

so

$$y' = -e^{-(x-\pi/4)}(c_1 \cos x + c_2 \sin x) + e^{-(x-\pi/4)}(-c_1 \sin x + c_2 \cos x).$$

Thus,

$$\sqrt{2}\,\frac{c_1}{2} + \sqrt{2}\,\frac{c_2}{2} = 2$$

and $-\sqrt{2}c_1 = -2$ and hence

$$y = \sqrt{2} e^{-(x-\pi/4)}(\cos x + \sin x).$$

11)

a) According the notation we used in the lecture, for the given D.E.

$$p(x) = x, \quad q(x) = e^{-x^2} > 0, \quad \text{for} \quad -\infty < x < \infty$$

Then

$$\frac{q' + 2pq}{2q^{3/2}} = 0$$

Hence the D.E. can be transformed into an equation with constant coefficients by letting

$$z = u(x) = \int q^{1/2} dx = \int e^{-x^2/2} dx$$

Substituting $z = u(x)$ in the D.E., we obtain the following D.E. for $y(z)$:

$$\frac{d^2y}{dz^2} - y = 0.$$

The general solution of the given D.E. is $y = c_1 \cos z + c_2 \sin z$ where $z = \int e^{-x^2/2} dx$.

b) In this case

$$p(x) = \frac{x^2 - 1}{x}, \quad q(x) = x^2 > 0, \quad \text{for} \quad -\infty < x < \infty$$

Then,

$$\frac{q' + 2pq}{2q^{3/2}} = 1$$

Hence the D.E. can be transformed into an equation with constant coefficients by letting

$$z = u(x) = \int q^{1/2} dx = \int x dx = \frac{x^2}{2}$$

Then the D.E. in $y(z)$ is

$$\frac{d^2y}{dz^2} + \frac{dy}{dz} + y = 0.$$

The solution of the above D.E. is

$$y(z) = e^{-z/2} \left(c_1 \cos \frac{\sqrt{3}}{2} z + c_2 \sin \frac{\sqrt{3}}{2} z \right).$$

Therefore, the solution of the given D.E. is

$$y(z) = e^{-x^2/4} \left(c_1 \cos \frac{\sqrt{3}}{4} x^2 + c_2 \sin \frac{\sqrt{3}}{4} x^2 \right).$$

12) Write the given D.E. in normal form, the

$$p(x) = \frac{\alpha}{x}, \quad q(x) = \frac{\beta}{x^2}$$

Thus

$$z = \int \left(\frac{1}{x^2} \right)^{1/2} dx = \ln x$$

will transform the given D.E. into

$$\frac{d^2y}{dz^2} + (\alpha - 1) \frac{dy}{dz} + \beta y = 0.$$

Note that sice β is constant, it can be neglected in defining z (z in general contains an arbitrary integration constant).

a) If we let $z = \ln x$ then we have

$$\frac{d^2y}{dz^2} + y = 0.$$

Since, $\alpha = 1$ and $\beta = 1$. Thus, $y = c_1 \cos z + c_2 \sin z$ with $z = \ln x, \quad x > 0$.

b) Let $z = \ln x$ and since $\alpha = -4$ and $\beta = -6$, then we have

$$\frac{d^2y}{dz^2} - 5\frac{dy}{dz} - 6y = 0.$$

The general solution of the above equation is

$$y = c_1 e^{-z} + c_2 e^{6z}, \qquad c_1, c_2 = \text{constant}$$

Hence the general solution of the given Euler's equation is

$$y = c_1 x^{-1} + c_2 x^6.$$