BİLKENT UNIVERSITY
Department of Mathematics

MATH 240, DIFFERENTIAL EQUATIONS, Solution of Homework set[1] # 2

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EXACT EQUATIONS

1)

a) $M(x,y) = 3x^2 - 2xy + 2$ and $N(x,y) = 6y^2 - x^2 + 3$ so $M_y = N_x = -2x$ and thus the given differential equation is exact. Integrate $M(x,y)$ w.r.t. x

$$F(x,y) = \int M(x,y)dx = x^3 - x^2 + 2x + g(y).$$

Differentiate $F(x,y)$ w.r.t. y and equate to $N(x,y)$. Then we have

$$-x^2 + g'(y) = 6y^2 - x^2 + 3$$

So, $g'(y) = 6y^2 + 3$ and $g(y) = 2y^3 + 3y$. Substitute $g(y)$ into $F(x,y)$ and recall that the equation which defines $y(x)$ implicitly is $F(x,y) = c, \quad c = \text{constant}$. Thus

$$x^3 - x^2y + 2x + 2y^3 + 3y = c$$

is the equation that yields the solution.

b) Write the equation in the form of $M(x,y)dx + N(x,y)dy = 0$. So, $M(x,y) = ax + by$ and $N(x,y) = bx + cy$. Since, $M_y = N_x = b$ the equation is exact. Integration of M w.r.t. x gives

$$F(x,y) = \frac{a}{2}x^2 + bxy + g(y)$$

Differentiate $F(x,y)$ w.r.t. y and equate to $N(x,y)$. Then we have $g'(y) = cy$ and $g(y) = \frac{c}{2}y^2$. Hence the solution is given by

$$\frac{a}{2}x^2 + bxy + \frac{c}{2}y^2 = k, \qquad k = \text{constant}.$$

c) As long as $x^2 + y^2 \neq 0$, we can simplify the equation by multiplying both sides by $(x^2 + y^2)^{3/2}$. We obtain an exact equation $xdx + ydy = 0$. The solution is given by $x^2 + y^2 = c, \quad c = \text{constant}$

2) Write the equation as $M(x)dx + N(y)dy = 0$. The exactness condition gives $M_y = N_x = 0$, so a separable equation is exact.

3)

a) The equation is not exact so lets find the integrating factor to make it an equation exact. Since,

$$\frac{1}{N}(M_y - N_x) = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} = 3$$

is a function of x alone there is a integrating factor depending only on x. Then the equation for the integrating factor $\mu(x)$ is

$$\frac{d\mu}{dx} = 3\mu$$

and the integrating factor is $\mu = e^{3x}$. If we multiply the given differential equation by $\mu(x)$ we get an exact equation and solve as in the previous examples.

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. Please check all the calculations/answers and notify me if you find any errors and/or misprints.

Ans.: $(3x^2y + y^2)e^{3x} = c, \qquad c = \text{constant}.$
b) Integrating factor μ is a function of x only, and one finds that $\mu = e^{-x}$. Note that the equation is also a linear first order equation for y
Ans.: $y = ce^x + 1 + e^{2x}, \qquad c = \text{constant}.$
c) Integrating factor μ is a function of y only, and $\mu(y) = y$. If we multiply the equation by y we obtain $ydx + (x - y\sin y)dy = 0$ which exact equation and can be solve as in the previous examples. Alternatively, if we rewrite the exact equation (last eq.) as $ydx + xdy = y\sin y\ dy$ which can be written as $d(xy) = y\sin y\ dy$. Then we can integrate both sides and get

$$xy + y\cos y - \sin y = c, \qquad c = \text{constant}$$

HOMOGENEOUS EQUATIONS
3)
a) The M and N in the equation are of homogenous of degree two in x and y. If we simplify the R.H.S. of the equation we get;

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2.$$

The substitution $y = vx$ leads to

$$v + x\frac{dv}{dx} = 1 + v + v^2,$$

which is separable. i.e.

$$\frac{dv}{1+v^2} = \frac{dx}{x}$$

Integrating, we get $\arctan v = \ln|x| + c, \quad c = \text{constant}$. Substituting for v we obtain

$$\arctan(y/x) = \ln|x| + c, \quad c = \text{constant}.$$

b) Dividing the numerator and denominator of the R.H.S. by x and substituting $y = vx$, we get

$$v + x\frac{dv}{dx} = \frac{4v-3}{2-v}$$

Separating the variables v and x yields

$$\frac{dx}{x} = \frac{2-v}{(v+3)(v-1)}.$$

Applying a partial fraction decomposition and integrating both sides, we find

$$\frac{1}{4}\ln|v-1| - \frac{5}{4}\ln|v+3| = \ln|x| + c, \quad c = \text{constant}.$$

Substitute $v = y/x$, perform some algebraic manipulations and get

$$|y - x| = c|y + 3x|, \qquad c = \text{constant}.$$

c) Homogenous equation, so let $v = y/x$, the we get the following separable differential equation for $v(x)$

$$\frac{v}{1+v^2}dv = \frac{1}{2x}dx.$$

Integrating once and substituting $v = y/x$ lead to

$$x^2 + y^2 = cx^3, \qquad c = \text{constant}.$$

4) By introducing new variable Y and X such that $y = Y + h$ and $x = X + k$ where $h, k =$constant, we get the following differential equation

$$\frac{dY}{dX} = -\frac{4X + 3Y + (15 + 4k + 3h)}{2X + Y + (7 + 2k + h)}$$

Letting $15 + 4k + 3h = 0$ and $7 + 2k + h = 0$ give $h = -1$ and $k = -3$. For these values of h, k the above equation for $Y(X)$ is homogenous. Introducing $v = Y/X$ (or $v = X/Y$) in the equation gives the separable equation for v.
Ans: $|x + y + 4||4x + y + 13|^2 = c, \qquad c =$ constant.

5) To show that $\mu(x, y)$ is an integrating factor, multiply the differential equation by the given $\mu(x, y)$. Since the differential equation is homogenous, we know that

$$\frac{M(x.y)}{N(x, y)} = F(y/x)$$

and thus the differential equation may be written as

$$\frac{F}{xF + y} dx + \frac{1}{xF + y} dy = 0$$

after multiplying both numerator and denominator by $1/N$ and substituting F for M/N. Now, you can check that

$$\frac{\partial}{\partial x}\left(\frac{1}{xF + y}\right) = \frac{\partial}{\partial y}\left(\frac{F}{xF + y}\right),$$

and hence the given differential equation is exact.

MISCELLANEOUS PROBLEMS
6)
a) Linear first order equation.
b) Linear first order equation in u.
c) Integrating factor depends on x only.
d) Separable equation.
e) Homogenous equation.
f) Linear first order equation in x.
g) Integrating factor depends on x only.
h) Integrating factor depends on y only.
k) Separable equation.

7) If we let $y = y_1 + \frac{1}{v(x)}$ where $y_1 = 1/x$, then we get the following linear first order equation in $v(x)$:

$$\frac{dv}{dx} + \frac{1}{x} v = -1, \qquad x \neq 0.$$

Its solution is

$$v = \frac{1}{x}\left(-\frac{1}{2}x^2 + c\right), \qquad c = \text{constant.}$$

Thus, the general solution of the given Riccati equation is

$$y_2(x) = \frac{1}{x} + \frac{2x}{2c - x^2}.$$