BİLKENT UNIVERSITY
Department of Mathematics

MATH 240, DIFFERENTIAL EQUATIONS, Solution of Homework set[1] # 1

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1)
a) Linear ODE.
Integrating Factor: $\mu(x) = e^{x^2}$
Solution: $y(x) = e^{-x^2}(x^2 + c), \quad c = \text{constant}.$
b) Linear ODE.
Integrating Factor: $\mu(x) = (1 + x^2)^2$
Solution: $y(x) = (1 + x^2)^{-2}(\tan^{-1} x + c), \quad c = \text{constant}.$
c) Linear ODE.
Integrating Factor: $\mu(x) = x^2$
Solution: $y(x) = x^{-2}(-x\cos x + \sin x + c), \quad c = \text{constant}.$

2)
a) Linear ODE.
Integrating Factor: $\mu(x) = xe^x$
Solution: $y(x) = x^{-1}e^{-x}(xe^x - e^x + c), \quad x \neq 0, \quad c = \text{constant}.$
Initial Condition (IC) gives $c = 2$
b) Linear ODE.
Integrating Factor: $\mu(x) = x^4$
Solution: $y(x) = -x^{-4}(xe^{-x} + e^{-x} + c), \quad x \neq 0, \quad c = \text{constant}.$
Initial Condition (IC) implies that $c = 0$
c) Linear ODE.
Integrating Factor: $\mu(x) = x^2$
Solution: $y(x) = x^{-2}(\sin x + c), \quad c = \text{constant}.$
Initial Condition (IC) implies that $c = 0$

3)
a) Linear ODE, solution exists and unique for all real x.
Integrating Factor: $\mu(x) = e^x$ and the solution is

$$y(x) = e^{-x}\left(\int_0^x \frac{e^t}{1+t^2}dt\right).$$

b) Linear ODE, solution exists for all $x \neq \pm 1$ and I.C. implies that the solution domain is $-1 < x < 1$.
Integrating Factor: $\mu(x) = (1 - x^2)^{1/2}$ and the solution is

$$y(x) = -\frac{1}{3}(1 - x^2) + \frac{7}{3}(1 - x^2)^{-1/2}.$$

c) Linear ODE, solution exists for all $x \neq -2, 0$ and I.C. implies that the solution domain is

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. Please check all the calculations/answers and notify me if you find any errors and/or misprints.

$-2 < x < 0$.
Integrating Factor: $\mu(x) = x^2 + 2x$ and the solution is

$$y(x) = \frac{x^3 + x + 1}{x(x+2)}$$

4) Bernoulli equation for $n = 3$. If we let $v = y^{-2}$ then we get the following linear equation for $v(x)$

$$\frac{dv}{dx} + 2(a\cos x + b)v = 2$$

which can be solved by introducing the integrating factor

$$\mu(x) = \int 2(a\cos x + b)dx.$$

Then the solution

$$v(x) = \frac{1}{\mu(x)}\left(2\int \mu(x)dx + c\right) \qquad c = \text{constant}.$$

Once $v(x)$ is known, one can find the solution of the given Bernoulli equation from $y = v^{-1/2}$.

5) Separable equation.

$$\int (3y^2 - 4)dy = 3\int x^2 dx + c, \qquad c = \text{constant}.$$

$$y^3 - 4y = x^3 - 1 + c$$

I.C. implies that $c = -1$. Use the existence and uniqueness theorem for $f(x,y) = 3x^2/(3y^2-4)$. $f(x,y)$ and $f_y(x,y)$ are continues for $y \neq \pm(4/3)^{1/2}$, then use the solution, and find the solution domain as

$$|x^3 - 1| < \frac{16}{3\sqrt{3}}$$

6)
a) Separable equation, and the solution is

$$xe^x - e^x + c = -\frac{y^2}{2}, \qquad c = \text{constant}.$$

I.C. implies that $c = 1/2$. $f(x,y) = -\dfrac{xe^x}{y}$ and $f_y(x,y)$ are continues for $y \neq 0$. Then the solution implies that $[2(1-x)e^x - 1]^(1/2) \neq 0$.
b) Separable equation, and the solution is

$$\frac{1}{3}\sin 3y = \frac{1}{2}\cos 2x + c, \qquad c = \text{constant}.$$

I.C. implies that $c = 1/2$. Then one can write the solution

$$\frac{1}{3}\sin 3y = \frac{1}{2}\cos 2x + \frac{1}{2} = \cos^2 x.$$

Solve the above equation for y and choose the branch that passes through the point $(\pi/2, \pi/3)$ and thus

$$y = \frac{\pi}{3} - \frac{1}{3}\arcsin(3\cos^2 x)$$

which is defined only $-1 \leq 3\cos^2 x \leq 1$ or $-\sqrt{1/3} \leq \cos x \leq \sqrt{1/3}$.
c) Separable equation, and the solution is

$$y = \tan(2x + x^2 + c), \qquad c = \text{constant}.$$

I.C. Implies that $c = 0$. Consider the initial point $x = 0$ and the graph of $\tan x$, so we have the following solution domain

$$\frac{-\pi}{2} < x^2 + 2x < \frac{\pi}{2}$$

7)
a) If $y' = f(x, y)$, in this case $f(x, y) = \frac{x-y}{2x+5y}$. By the existence and uniqueness theorem: $f(x, y)$ and f_y are continues everywhere except $2x+5y \neq 0$. Therefore, the region is $2x+5y > 0$ or $2x+5y < 0$.
b) In this case

$$f(x, y) = \frac{1 + x^2}{y(3 - y)}$$

$f(x, y)$ and f_y are continues everywhere except $y = 0$ and $y = 3$.
c) In this case

$$f(x, y) = (x^2 + y^2)^{3/2} \text{ and } f_y = 3y(x^2 + y^2)^{1/2}$$

$x^2 + y^2 > 0$ always ,so $f(x, y)$ and f_y are continues everywhere.

8)
a) Separable equation. Solution is

$$y^2 = -4x^2 + c, \qquad c = \text{constant}.$$

I.C. implies that $c = y_0^2$. Therefore the solution of the I.V.P. is

$$y = \pm\sqrt{y_0^2 - 4x^2}$$

In order to find the solution domain, one should set $y_0^2 - 4x^2 > 0$. Therefore, $|x| < |y_0|/2$.
b) Separable equation. Solution is

$$y = -\frac{1}{x^2 + c}, \qquad c = \text{constant}.$$

I.C. implies that $c = -1/y_0$ $y_0 \neq 0$. Therefore the solution of the I.V.P. is

$$y = \frac{y_0}{y_0 x^2 - 1}$$

Note that $y = 0$ is also solution if $y_0 = 0$. Therefore, if $y_0 = 0$ the interval is $-\infty < x < \infty$. If $y_0 > 0$, then $y_0 x^2 - 1 = 0$ yields $|x| < 1/\sqrt{y_0}$.
c) Separable equation. Solution is

$$y^{-2} = 2x + c, \qquad c = \text{constant}.$$

I.C. implies that $c = 1/y_0^2$, $y_0 \neq 0$. Therefore the solution of the I.V.P. is

$$y = \frac{y_0}{\sqrt{2y_0^2 x + 1}}, \qquad \text{if } y_0 \neq 0$$

Note that $y = 0$ is also solution if $y_0 = 0$. Therefore, if $y_0 = 0$ the interval is $-\infty < x < \infty$. If $y_0 \neq 0$, then $2y_0^2 x + 1 > 0$ yields $x > -1/(2y_0^2)$.

9)
Separable equation. Solution of the I.V.P.

$$y = \frac{2}{3}x^{3/2} \quad x \geq 0.$$

a) No
b) Note that

$$y = -\frac{2}{3}x^{3/2} \quad x \geq 0.$$

is also solution. Moreover $y = 0$ and for arbitrary x_0 $y = \pm\frac{2}{3}(x - x_0)^{3/2} \quad x \geq x_0$. are also solutions. Therefore if we choose $x_0 = 1/2$ we find the solution which passes through the point $(2, 1)$.
c)

$$|y| \leq \left(\frac{4}{3}\right)^{3/2}.$$