BİLKENT UNIVERSITY
Department of Mathematics

Date: 6 November 2012 NAME: ....**SOLUTION KEY**.............

Time: 18:00-20:00 STUDENT NO: ....

Fall 2012-13, A.Gheondea, A.Degtiarev, U.Muğan, **SECTION:** **01** (AG) **02** (AD) **03** (UM)

Math 240.01-03, Differential Equations, Midterm Exam # 1

1	2	3	4	5	TOTAL
20	20	20	20	20	100

(Do not write anything on the above table)

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1) Find the general solution of $(x^2 + xy + y^2)dx - x^2dy = 0$.

(20 points)

Homogenous type D.E. Let $v = y/x$, $x \neq 0$. The the given D.E. yields

$$v + xv' = 1 + v + v^2,$$

which is separable the D.E. Since

$$\frac{1}{1+v^2}\frac{dv}{dx} = \frac{1}{v}.$$

Integrating above equation gives:

$$\arctan v = \ln|x| + c, \qquad c = \text{constant}.$$

So the implicit solution of the given D.E. for $x \neq 0$ is,

$$\arctan\frac{y}{x} = \ln|x| + c.$$

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2) Find the general solution of $2xy\,dx + (2x^2 + 3y)dy = 0$.

(20 points)

The given D.E. can be transformed to an exact type D.E. by using an integrating factor $\mu = \mu(y)$.

$$\frac{M_y - N_x}{M} = \frac{1}{y},$$

then $\mu(y) = y$. Then there exists $F(x, y) = c$=constant, such that $F_x = \mu M$ and $F_y = \mu N$.

$$F_x = 2xy^2, \qquad F_y = 2x^2y + 3y^3.$$

Integrating the first equation gives

$$F(x, y) = x^2y^2 + h(y),$$

where $h(y)$ is an arbitrary function of y. By using $F_y = 2x^2y + 3y^3$, we obtain

$$h'(y) = 3y^3.$$

So, $h(y) = y^3$.

Therefore the implicit solution of the given D.E. is

$$F(x, y) = x^2y^2 + y^3 = c, \qquad c = \text{constant}.$$

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3) Solve the following initial value problem:

$$y'' + 2y(y')^3 = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

(20 points)

The solution is $y = 1$, and this is obvious. (To be on the safe side, you should also refer to the uniqueness theorem, which does apply to this equation, as it is of the form $y'' = f(t; y; y0)$.) If you want to do it "by the book", OK, here it is. Let $y' = v(y)$, so that $y'' = v\frac{dv}{dy}$.

The equation becomes

$$v\frac{dv}{dy} + 2yv^3 = 0,$$

hence

$$v = 0,$$

or

$$\frac{dv}{dy} + 2vy^2 = 0.$$

In the former case, $y' = 0$, hence $y =$constant and, using the initial values, $y = 1$. The latter equation is separable. When separating the variables (dividing by v^2), we observe again that $v = 0$ is a solution. Due to the uniqueness theorem, it is the only solution satisfying $v(1) = 0$. Hence, as before, $y = 1$. If you still want to take it further, you have:

$$\frac{dv}{v^2} = -2y\,dy, \quad \frac{1}{v} = y^2 + c, \quad v = \frac{1}{y^2 + c}.$$

Among these solutions, there is none satisfying $v(1) = 0$. (And, due to the uniqueness theorem, there should not be!)

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4) Find necessary and sufficient condition on the coefficient functions $M(x, y)$, and $N(x, y)$ for the differential equation $M(x, y)dx + N(x, y)dy = 0$ to have an integrating factor of the form $\mu(xy)$. Assume that M and N are continuously differentiable on the whole plane.

(20 points)

We want a function $\mu(t)$ such that the equation

$$\mu(xy)M(x, y)dx + \mu(xy)N(x, y)dy = 0,$$

is exact, i.e., $(\mu M)_y = (\mu N)_x$. Since, $\frac{\partial}{\partial x}\mu(xy) = y\mu'(xy)$, and $\frac{\partial}{\partial y}\mu(xy) = x\mu'(xy)$, (where $'$ stands for the ordinary univariate derivative), we have

$$x\mu' M + y\mu M_y = y\mu' N + \mu N_x.$$

hence,

$$\frac{\mu'}{\mu} = \frac{N_x - M_y}{xM - yN}.$$

The right hand side is a function of xy only. Hence, the latter equation has a solution if and only if the function

$$f(x, y) := \frac{N_x - M_y}{xM - yN},$$

depends on xy only. Well, if you want to make it even more formal, a function $f(x, y)$ depends on xy only if and only if $xf_x = yf_y$ (of course, assuming sufficient differentiability). Thus, the condition is

$$x\left(\frac{N_x - M_y}{xM - yN}\right)_x = y\left(\frac{N_x - M_y}{xM - yN}\right)_y.$$

NB: This computation (except the last part) merely mimics what is done in the textbook for the cases $\mu(x)$ and $\mu(y)$.

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5) Classify the the following differential equations.
e.g. Linear in y, Homogenous, Separable, Exact, ...
DO NOT find the solutions and write ONLY ONE answer in each box.

a) $(1 + x^2 + y^2 + x^2y^2)dy = y^2dx$	**Ans: Separable**
b) $x^2y' = x^2y^2 - xy - 4$	**Ans: Riccati**
c) $(x + y)^2dx + (2xy + x^2 - 1)dy = 0$	**Ans: Exact**
d) $(\sqrt{x} + \sqrt{y})^2dx = x\,dy$	**Ans: Homogenous**
e) $2\,\frac{dy}{dx} = \frac{y}{x} - \frac{x}{y^2}$	**Ans: Bernoulli, or Exact with** $\mu(x)$

($4 \times 5 = 20$ **points)**

You can use the space below for your own calculations