BİLKENT UNIVERSITY
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MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 8

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Homework problems from the 2nd Edition, SECTION 3.4

5(5))[2]

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -9 & 1 \\ -10 & 12 \end{bmatrix}, \qquad \begin{bmatrix} -4 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 8 \\ 11 & 5 \end{bmatrix}.$$

8(8))

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 15 \\ 34 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 0 \\ -1 & 4 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 9 \\ 7 & -20 & 13 \\ 16 & -25 & 38 \end{bmatrix}.$$

11(11))

$$\mathbf{AB} = \begin{bmatrix} 3 & -5 \end{bmatrix} \begin{bmatrix} 2 & 7 & 5 & 6 \\ -1 & 4 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 1 & 5 & 3 \end{bmatrix}.$$

But, the product **BA** is not defined.

15(15))

$$\mathbf{A}(\mathbf{BC}) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \left(\begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{bmatrix} 4 & 5 \end{bmatrix} = \begin{bmatrix} 12 & 15 \\ 8 & 10 \end{bmatrix}.$$

$$(\mathbf{AB})\mathbf{C} = \left(\begin{bmatrix} 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \right) \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 6 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 12 & 15 \\ 8 & 10 \end{bmatrix}.$$

17(17)) Since the coefficient matrix is already is in echelon form, so we can write down the solutions by the back substitution.

$$x_3 = s, \quad x_4 = t, \quad x_2 = -2s + 7t, \quad x_1 = 5s - 4t.$$

In vector form

$$\vec{x} = s(5, -2, 1, 0) + t(-4, 7, 0, 1).$$

21(21)) Since the coefficient matrix is already is in echelon form, so we can write down the solutions by the back substitution.

$$x_3 = r, \quad x_4 = s, \quad x_5 = t, \quad x_2 = -2r + 3s - 4t, \quad x_1 = r - 2s - 7t.$$

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

[2] The number in the parenthesis denotes the problem number in the International Edition of the textbook

In vector form

$$\vec{x} = r(1, -2, 1, 0, 0) + s(-2, 3, 0, 1, 0) + t(-7, -4, 0, 0, 1).$$

25(25)) The matrix equation

$$\begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

leads to four scalar equations

$$\begin{aligned} 5a + 7c &= 1 \\ 2a + 3c &= 0 \\ 5b + 7d &= 0 \\ 2b + 3d &= 1 \end{aligned}$$

that we can solve for $a = 3, \quad b = -7, \quad c = 2, \quad d = 5$. Hence the inverse of $\mathbf{A}$, such that $\mathbf{AB} = \mathbf{I}$, is

$$\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}.$$

We also find $\mathbf{BA} = \mathbf{I}$.

26(26)) The matrix equation

$$\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

leads to four scalar equations

$$\begin{aligned} a - 2c &= 1 \\ -2a + 4c &= 0 \\ b - 2d &= 0 \\ -2b + 4d &= 1 \end{aligned}$$

But the two equations in a and c are inconsistent, because $(-1)(1) \neq 0$. Similarly, the two equations in b and d are inconsistent. So the given matrix $\mathbf{A}$ is not invertible.

29(29))

$$(a+d)\mathbf{A} - (ad - bc)\mathbf{I} = (a+d) \begin{bmatrix} a & b \\ c & d \end{bmatrix} - (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (a^2 + ad) - (ad - bc) & ab + bd \\ ac + cd & (ad + d^2) - (ad - bc) \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \mathbf{A}^2.$$

30(30)) Since

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

then $a + d = 4$ and $ad - bc = 3$. Hence,

$$\mathbf{A}^2 = 4\mathbf{A} - 3\mathbf{I} = 4 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$\mathbf{A}^3 = 4\mathbf{A}^2 - 3\mathbf{A} = 4\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} - 3\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix}$$

$$\mathbf{A}^4 = 4\mathbf{A}^3 - 3\mathbf{A}^2 = 4\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix} - 3\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 41 & 40 \\ 40 & 41 \end{bmatrix}$$

$$\mathbf{A}^5 = 4\mathbf{A}^4 - 3\mathbf{A}^3 = 4\begin{bmatrix} 41 & 40 \\ 40 & 41 \end{bmatrix} - 3\begin{bmatrix} 14 & 13 \\ 13 & 14 \end{bmatrix} = \begin{bmatrix} 122 & 121 \\ 121 & 122 \end{bmatrix}$$

31(31)) a) Let

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$$

then

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \begin{bmatrix} 3 & 4 \\ -1 & 10 \end{bmatrix}\begin{bmatrix} 1 & -6 \\ -7 & -4 \end{bmatrix} = \begin{bmatrix} -25 & -34 \\ -71 & -34 \end{bmatrix}.$$

On the other hand,

$$\mathbf{A}^2 - \mathbf{B}^2 = \begin{bmatrix} 8 & -5 \\ -20 & 13 \end{bmatrix} - \begin{bmatrix} 16 & 40 \\ 24 & 64 \end{bmatrix} = \begin{bmatrix} -8 & -45 \\ -44 & -51 \end{bmatrix}.$$

b) If $\mathbf{AB} = \mathbf{BA}$ then

$$(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B}) = \mathbf{A}(\mathbf{A}-\mathbf{B}) + \mathbf{B}(\mathbf{A}-\mathbf{B}) = \mathbf{A}^2 - \mathbf{AB} + \mathbf{BA} - \mathbf{B}^2 = \mathbf{A}^2 - \mathbf{B}^2.$$

33(33)) Four different 2×2 matrices $\mathbf{A}$ with $\mathbf{A}^2 = \mathbf{I}$ are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

34(34)) If

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \neq 0$$

then $\mathbf{A}^2 = (0)\mathbf{A} - (0)\mathbf{I} = 0$.

35(35)) If

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \neq 0$$

then $\mathbf{A}^2 = (1)\mathbf{A} - (0)\mathbf{I} = \mathbf{A}$.

37(37)) If

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \neq 0$$

then $\mathbf{A}^2 = (0)\mathbf{A} - (1)\mathbf{I} = -\mathbf{I}$.