BİLKENT UNIVERSITY
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MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 7

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Homework problems from the 2nd Edition, SECTION 3.3

3(3))[2] Let

$$\mathbf{A} = \begin{bmatrix} 3 & 7 & 15 \\ 2 & 5 & 11 \end{bmatrix}$$

Do the following row operations:

$$\begin{bmatrix} 3 & 7 & 15 \\ 2 & 5 & 11 \end{bmatrix}, (-R_2+R_1) \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 11 \end{bmatrix}, (-2R_1+R_2) \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{bmatrix}, (-2R_2+R_1) \rightarrow \mathbf{E} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}.$$

NOTE THAT: The reduced row-echelon form of a matrix is **UNIQUE**, so different sequence of elementary row operations should produce the same reduced row-echelon form of the given matrix.

9(9)) Let

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 18 \\ 0 & 1 & 4 \\ 4 & 1 & 12 \end{bmatrix}$$

Do the following row operations: $R_1 - R_3$, $R_3 - 4R_1$, $R_3 + 3R_2$, and $R_1 - R_2$ and get

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

NOTE THAT: The reduced row-echelon form of a matrix is **UNIQUE**, so different sequence of elementary row operations should produce the same reduced row-echelon form of the given matrix.

13(13)) Let

$$\mathbf{A} = \begin{bmatrix} 2 & 7 & 4 & 0 \\ 1 & 3 & 2 & 1 \\ 2 & 6 & 5 & 4 \end{bmatrix}$$

Do the following row operations: Swap(R_1, R_2), $R_2 - 2R_1$, $R_1 - 3R_2$, and so on, and get

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

NOTE THAT: The reduced row-echelon form of a matrix is **UNIQUE**, so different sequence of

[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

[2] The number in the parenthesis denotes the problem number in the International Edition of the textbook

elementary row operations should produce the same reduced row-echelon form of the given matrix.

16(16)) Let

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 15 & 7 \\ 2 & 4 & 22 & 8 \\ 2 & 7 & 34 & 17 \end{bmatrix}$$

Do the following row operations: $R_2 - 2R_1$, $R_3 - 2R_1$, $(-1/2)R_2$, $R_3 - R_2$ and so on, and get

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

NOTE THAT: The reduced row-echelon form of a matrix is **UNIQUE**, so different sequence of elementary row operations should produce the same reduced row-echelon form of the given matrix.

33(33)) If the upper left element of a 2×2 reduced echelon matrix is 1, then the possibilities are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$$

depending on whether there is a nonzero element in the second row. If the upper left element is zero, so both elements of the second row are also 0, the possibilities are

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

34(34)) If the upper left element of 3×3 reduced echelon matrix is 1 then the possibilities are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

depending on whether the second and third row contain any nonzero elements. If the upper left element is zero, so the first column and third row contain no nonzero element, then use of the four 2×2 reduced echelon matrices of problem # 33 (for the upper right 2×2 submatrix for our reduced 3×3 matrix) gives the additional possibilities

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

35(35)) (a) If (x_0, y_0) is a solution, then it follows that

$$a(kx_0) + b(ky_0) = k(ax_0 + by_0) = k.0 = 0, \quad c(kx_0) + d(ky_0) = k(cx_0 + dy_0) = k.0 = 0,$$

so (kx_0, ky_0) is also a solution.

(b) If (x_1, y_1) and (x_2, y_2) are solutions, then it follows that is also a solution.

$$a(x_1 + x_2) + b(y_1 + y_2) = (ax_1 + by_1) + (ax_2 + by_2) = 0 + 0 = 0,$$

$$c(x_1+x_2)+d(y_1+y_2)=(cx_1+dy_1)+(cx_2+dy_2)=0+0=0,$$

so (x_1+x_2, y_1+y_2) is also a solution.

36(36)) By the problem #32, the coefficient matrix of the given homogenous system is row equivalent to the 2×2 identity matrix. Therefore by the theorem, (th'm 4 in the text) the given system has only the trivial solution.

39(39)) The augmented coefficient matrix of the given system is

$$\mathbf{A}=\begin{bmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ pa_1+qa_2 & pb_1+qb_2 & pc_1+qc_2 & 0 \end{bmatrix}$$

If we do the following row operation: $R_3-(pR_1+qR_2)$ we obtain

$$\begin{bmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

corresponding to two homogenous linear equations in three unknowns. Hence there is at least one free variable, and thus the system has a nontrivial family of solutions.

40(40)) In reducing further from the echelon matrix $\mathbf{E}$ to the matrix $\mathbf{E}^*$, the leading entries of $\mathbf{E}$ become the leading ones in the reduced echelon matrix $\mathbf{E}^*$. Thus the nonzero rows of $\mathbf{E}^*$ come precisely from the nonzero rows of $\mathbf{E}$. We therefore are talking about the same rows and in particular about the same *number* of rows in either case.