BİLKENT UNIVERSITY
Department of Mathematics

MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 4

U. Muğan June 20, 2008

Homework problems from the 2nd Edition, SECTION 1.6

3 (3))[2] Let $v = y/x$ then D.E. leads to following separable D.E.

$$xv' = 2v^{1/2}$$

then

$$\int \frac{1}{2}v^{-1/2}dv = \int \frac{dx}{x}, \qquad v^{1/2} = \ln x + c, \qquad c = \text{constant}$$

Therefore the solution is

$$y(x) = x(\ln x + c)^2.$$

Note that the given D.E. is of homogenous type.

9 (9)) If we let $v = y/x$ then D.E. leads to following separable D.E.

$$xv' = v^2$$

then

$$\int \frac{dv}{v^2} = \int \frac{dx}{x}, \qquad v^{-1} = -\ln x + c, \qquad c = \text{constant}$$

Therefore the solution is

$$x = y(c - \ln x).$$

10 (10)) Let $v = y/x$ then D.E. leads to following separable D.E.

$$xvv' = 2v^2 + 1, \qquad \int \frac{4vdv}{2v^2 + 1} = \int \frac{4dx}{x}$$

$$\ln(2v^2 + 1) = c\ln x + \ln c, \qquad c = \text{constant}$$

The solution of the given D.E. is

$$x^2 + 2y^2 = cx^6.$$

12 (12)) The given D.E. is of homogenous type, so let $v = y/x$, then D.E. leads to following separable D.E.

$$xvv' = (v^2 + 4)^{1/2}$$

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

[2] The number in the parenthesis denotes the problem number in the International Edition of the textbook

$$\int \frac{vdv}{(v^2+4)^{1/2}} = \int \frac{dx}{x}, \qquad (v^2+4)^{1/2} = \ln x + c, \qquad c = \text{constant}$$

Hence the solution of the given D.E. is

$$4x^2 + y^2 = x^2(\ln x + c)^2.$$

18 **(18)**) Let $v = x + y$ then the given D.E. leads to following separable D.E.

$$vv' = v + 1, \qquad x = \int \frac{vdv}{v+1} = \int \left[1 - \frac{1}{v+1}\right] dv = v - \ln(v+1) + c,$$

where c is an integration constant. Then the solution of the given D.E. is

$$y(x) = \ln(x + y + 1) + c.$$

24 **(24)**) Let $v = 1/y^2$, then we have the following first order linear D.E. for v

$$v' + 2v = \frac{e^{-2x}}{x}$$

The integrating factor ρ is

$$\rho(x) = \exp\left[\int 2dx\right] = e^{2x}.$$

and the solution of the given D.E. is

$$y^2 = \frac{e^{2x}}{c + \ln x}, \qquad c = \text{constant}.$$

29 **(29)**) Substitution $v = \sin y$ yields the homogenous equation

$$2xvv' = 4x^2 + v^2$$

Then let $u = v/x$ and obtain a separable equation. The solution of the given D.E. is

$$\sin^2 y = 4x^2 - cx, \qquad c = \text{constant}.$$

32 **(32)**) Since

$$M(x,y) = 4x - y, \qquad N(x,y) = 6y - x$$

and $M_y = N_x$ the given D.E. is exact. Therefore there exits $F(x,y)$ such that $F_x = M$, $F_y = N$ and the solution of the D.E. is $F(x,y) = c$, $c =$constant.

$$F = \int (4x - y)dx = 2x^2 - xy + g(y), \qquad g(y) \ \text{ is an arbitrary function of } y.$$

$$F_y = -x + g'(y) = N = 6y - x$$
$$g'(y) = 6y, \qquad g(y) = 3y^2$$

Therefore the solution of the D.E. is

$$2x^2 - xy + 3y^2 = c, \qquad c = \text{constant}.$$

36 (36)) The given D.E. is exact, so

$$F = \int (1 + ye^{xy})dx = x + e^{xy} + g(y), \qquad g(y) \text{ is an arbitrary function of } y.$$

$$F_y = xe^{xy} + g'(y) = N = 2y + xe^{xy}$$

$$g'(y) = 2y, \quad g(y) = y^2$$

Therefore the solution of the D.E. is

$$x + e^{xy} + y^2 = c, \qquad c = \text{constant}.$$

39 (39)) The given D.E. is exact, so

$$F = \int (3x^2y^3 + y^4)dx = x^3y^3 + xy^4 + g(y), \qquad g(y) \text{ is an arbitrary function of } y.$$

$$F_y = 3x^3y^2 + 4xy^3 + g'(y) = N = 3x^3y^2 + y^4 + 4xy^3$$

$$g'(y) = y^4, \quad g(y) = y^5/5$$

Therefore the solution of the D.E. is

$$x^3y^3 + xy^4 + \frac{1}{5}y^5 = c, \qquad c = \text{constant}.$$

44) Since the given D.E. autonomous type, let $y' = v$ then

$$y'' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx}.$$

then the given D.E. yields

$$yvv' + v^2 = 0$$

and for $v \neq 0$ we get the following separable equation

$$yv' = -v.$$

The solution of the above D.E. is $\ln v = -\ln y + \ln c_1$ or $v = c_1/y$ where c_1 is an integration constant. Then

$$x(y) = \int \frac{dx}{dy}\, dy = \int \frac{1}{v}\, dy = \int \frac{y}{c_1}\, dy, \qquad \text{for } c_1 \neq 0,$$

$$x(y) = \frac{y^2}{2c_1} + c_2, \qquad c_2 = \text{constant}$$

or

$$x(y) = Ay^2 + B, \qquad A, B = \text{constant}$$

50) Since y does nor appear in the D.E. let $y' = v$, the $y'' = v'$ and the D.E. gives the following first order O.D.E.

$$v' = (x + v)^2.$$

If we let $u = x + v$ then the D.E. for v yields the following separable equation for u

$$u' - 1 = u^2.$$

Then

$$\int \frac{du}{1+u^2} = \int dx, \quad \Rightarrow \quad \tan^{-1} u = x + c_1, \quad c_1 = \text{constant},$$

$$u = x + y' = \tan(x + c_1)$$

Therefore

$$y' = \tan(x + c_1) - x$$

By integrating the above equation, we get the solution y of the given D.E.

$$y(x) = \ln|\sec(x + c_1)| - \frac{1}{2}x^2 + c_2, \quad c_1 = \text{constant}.$$

53) (For the detail see the solution of the problem # 44) Since the given D.E. autonomous type, let $y' = v$. Then $y'' = vv' = v\frac{dv}{dy}$. Then the given D.E. yields

$$vv' = 2yv, \quad \Rightarrow \quad \int dv = 2\int y dy, \quad \Rightarrow \quad v = y^2 + c_1, \quad c_1 = \text{constant}.$$

Then

$$x = \int \frac{1}{v}\, dy = \int \frac{dy}{y^2 + c_1} = \frac{1}{c}\tan^{-1}\frac{y}{c} + c_2,$$

where $c = \sqrt{c_1}$ and c_2 is an integration constant.

$$\tan^{-1}\frac{y}{c} = c(x - c_2), \quad \Rightarrow \quad \frac{y}{c} = \tan(cx - cc_2).$$

Hence the solution of the given D.E. is

$$y(x) = c\tan(cx + k),$$

where $k = cc_2$.

54) Since the given D.E. autonomous type, let $y' = v$ then

$$y'' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\,\frac{dv}{dy}.$$

then the given D.E. yields

$$y\,v\,\frac{dv}{dy} - 3v^2 = 0, \quad \Rightarrow \quad \int \frac{dv}{v} = \int \frac{3dy}{y}.$$

$$\ln v = 3\ln y + \ln c_1, \quad \Rightarrow \quad v = c_1 y^3, \quad c_1 = \text{constant}.$$

Hence

$$x(y) = \int \frac{dx}{dy}\, dy = \int \frac{1}{v}\, dy = \int \frac{1}{c_1 y^3}\, dy = -\frac{1}{2c_1 y^2} + c_2.$$

Therefore implicit solution of the given D.E. is

$$cy^2(c_2 - x) = 1,$$

where $c = 2c_1$.

58 (46)) The substitution $v = \ln y$ into D.E. the we obtain the following linear equation for v,

$$xv' + 2v = 4x^2$$

Then the integrating factor is $\rho = x^2$ and the solution of the given D.E. is

$$y(x) = \exp\left(x^2 + \frac{c}{x^2}\right), \qquad c = \text{constant}.$$

60 **(48)**) Substitution $x = u + 3$ and $y = v - 2$ yields the following homogenous type D.E.

$$\frac{dv}{du} = \frac{-u + 2v}{4u - 3v}.$$

The substitution $p = v/u$ leads to

$$\ln u = \int \frac{(4 - 3p)dp}{(3p + 1)(p - 1)} = \frac{1}{4} \int \left[\frac{1}{p - 1} - \frac{15}{3p + 1}\right] dp$$

$$\ln u = \frac{1}{4}[\ln(p - 1) - 5\ln(3p + 1) + \ln c], \qquad c = \text{constant}.$$

$$u^4 = \frac{c(p - 1)}{(3p + 1)^5} = \frac{c[(v/u) - 1]}{[3(v/u) + 1]^5} = \frac{cu^4(v - u)}{(3v + u)^5}$$

Hence the implicit solution of the given D.E. is

$$(x + 3y + 3)^5 = c(y - x + 5).$$

64 **(52)**) Substitution $y = x + \frac{1}{v}$ yields the following linear D.E.

$$v' - 2xv = 1$$

The integrating factor for the above D.E. is

$$\rho(x) = e^{-x^2}.$$

The general solution of the linear D.E. for v can be written in terms of the error function $\text{erf}(x)$ (see Section 1.5, problem 29).

$$v(x) = e^{x^2}\left[c + \frac{\sqrt{\pi}}{2}\,\text{erf}(x)\right], \qquad c = \text{constant}.$$

Hence the general solution of the given Riccati equation is

$$y(x) = x + e^{-x^2}\left[c + \frac{\sqrt{\pi}}{2}\,\text{erf}(x)\right]^{-1}.$$

65 **(53)**) Substitute $y = x + \frac{1}{v}$ and get the following simple D.E. $v' = -1$. Then $v = c - x$ where c is an integration constant. Therefore the general solution of the given Riccati equation is

$$y(x) = x + \frac{1}{c - x}.$$