BİLKENT UNIVERSITY
Department of Mathematics

MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 3

U. Muğan June 20, 2008

Homework problems from the 2nd Edition, SECTION 1.5

5 (5))[2] Linear first order D.E. which can be solved by introducing the integrating factor $\rho(x)$

$$\rho = \exp\left[\int \frac{2}{x}\,dx\right] = e^{2\ln x} = x^2.$$

The D.E. can be written as

$$\frac{d}{dx}(yx^2) = 3x^2$$

Integration yields

$$y(x) = x + \frac{c}{x^2}, \qquad c = \text{constant}$$

I.C. implies that $c = 4$, so the solution of the I.V.P. is

$$y(x) = x + \frac{4}{x^2}.$$

10 (10)) Integrating factor is

$$\rho(x) = \exp\left[\int -\frac{3}{2x}\,dx\right] = e^{(-3\ln x)/2} = x^{-3/2}.$$

Then the D.E. can be written as

$$\frac{d}{dx}(yx^{-3/2}) = \frac{9}{2}x^{1/2}$$

Therefore the general solution of the D.E. is

$$y(x) = 3x^3 + cx^{3/2}, \qquad c = \text{constant}.$$

14 (14)) Integrating factor is

$$\rho(x) = \exp\left[\int -\frac{3}{x}\,dx\right] = e^{-3\ln x} = x^{-3}.$$

Then the D.E. can be written as

$$\frac{d}{dx}(yx^{-3}) = \frac{1}{x}$$

---

[1]I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

[2]The number in the parenthesis denotes the problem number in the International Edition

Therefore the general solution of the D.E. is

$$y(x) = x^3 \ln x + cx^3, \qquad c = \text{constant}.$$

I.C. implies that $c = 10$, so the solution of the I.V.P. is

$$y(x) = x^3 \ln x + 10x^3.$$

20 **(20)**) Integrating factor is

$$\rho(x) = \exp\left[\int (-1 - x)\, dx\right] = e^{-x-\frac{x^2}{2}}.$$

Then the D.E. can be written as

$$\frac{d}{dx}(ye^{-x-\frac{x^2}{2}}) = (1 + x)e^{-x-\frac{x^2}{2}}$$

Therefore the general solution of the D.E. is

$$y(x) = -1 + ce^{x+\frac{x^2}{2}}, \qquad c = \text{constant}.$$

I.C. implies that $c = 1$, so the solution of the I.V.P. is

$$y(x) = -1 + e^{x+\frac{x^2}{2}}.$$

24 **(24)**) Integrating factor is

$$\rho(x) = \exp\left[\int \frac{3x}{x^2 + 4}\, dx\right] = e^{3\ln(x^2+4)/2} = (x^2 + 4)^{3/2}.$$

Then the D.E. can be written as

$$\frac{d}{dx}(y(x^2 + 4)^{3/2}) = x(x^2 + 4)^{1/2}$$

Therefore the general solution of the D.E. is

$$y(x) = \frac{1}{3} + c(x^2 + 4)^{-3/2}, \qquad c = \text{constant}.$$

I.C. implies that $c = 16/3$, so the solution of the I.V.P. is

$$y(x) = \frac{1}{3} + \frac{16}{3}(x^2 + 4)^{-3/2}.$$

26 **(26)**) If we interchange the role of the variables, i.e. if we let x be dependent variable and y be independent variable, the the D.E. takes the form of

$$y^3 \frac{dx}{dy} + 4y^2 x = 1.$$

Then the integrating factor is

$$\rho(y) = \exp\left[\int \frac{4}{y}\, dy\right] = e^{4\ln y} = y^4.$$

Then the D.E. can be written as

$$\frac{d}{dy}\left[xy^4\right] = y$$

Therefore the general solution of the D.E. is

$$x(y) = \frac{1}{2y^2} + \frac{c}{y^4}, \qquad c = \text{constant}.$$

27 (**27**)) If we interchange the role of the variables, i.e. if we let x be dependent variable and y be independent variable, the the D.E. takes the form of

$$\frac{dx}{dy} - x = ye^y.$$

Then the integrating factor is

$$\rho(y) = \exp\left[-\int dy\right] = e^{-y}.$$

Then the D.E. can be written as

$$\frac{d}{dy}\left[xe^{-y}\right] = y$$

Therefore the general solution of the D.E. is

$$x(y) = \left[\frac{1}{2}y^2 + c\right]e^y, \qquad c = \text{constant}.$$

28 (**28**)) If we interchange the role of the variables, i.e. if we let x be dependent variable and y be independent variable, the the D.E. takes the form of

$$(1+y^2)\frac{dx}{dy} - 2yx = 1.$$

Then the integrating factor is

$$\rho(y) = \exp\left[\int \frac{-2y}{1+y^2}\,dy\right] = e^{-\ln(y^2+1)} = \frac{1}{1+y^2}.$$

Then the D.E. can be written as

$$\frac{d}{dy}\left[\frac{x}{1+y^2}\right] = \frac{1}{(1+y^2)^2}$$

Using the integrating table or trigonometric substitution yields

$$\frac{x}{1+y^2} = \int \frac{dy}{(1+y^2)^2} = \frac{1}{2}\left[\frac{y}{1+y^2} + \tan^{-1} y + c\right], \qquad c = \text{constant}.$$

Therefore the general solution of the D.E. is

$$x(y) = \frac{1}{2}\left[y + (1+y^2)(\tan^{-1} y + c)\right].$$

30 (**30**)) If we divide the D.E. by $2x$ and multiply the D.E. by the integrating factor $\rho(x) = x^{-1/2}$ then we get the following D.E.

$$x^{-1/2}y' - \frac{1}{2}x^{-3/2}y = x^{-1/2}\cos x.$$

which can be written as

$$\frac{d}{d}\left[x^{-1/2}y\right] = x^{-1/2}\cos x.$$

If we integrate both sides of the above equation, we get

$$x^{-1/2}y = c + \int_1^x t^{-1/2}\cos t\,dt, \qquad c = \text{constant}.$$

The I.C. implies that $c = 0$, therefore the particular solution is

$$y(x) = x^{1/2}\int_1^x t^{-1/2}\cos t\,dt.$$

32 **(32))**
a) If $y = A\cos x + B\sin x$ then

$$y' + y = (A+B)\cos x + (B-A)\sin x = 2\sin x$$

provided that $A = -1$ and $B = 1$. These coefficients give the particular solution

$$y_p(x) = \sin x - \cos x.$$

b) The general solution of the equation $y' + y = 0$ is $y(x) = ce^{-x}$, $c =$constant, so addition to the particular solution found in the first part gives

$$y(x) = ce^{-x} + \sin x - \cos x.$$

c) The I.C. implies that $c = 2$, so the particular solution is

$$y(x) = 2e^{-x} + \sin x - \cos x.$$

33 **(33))** **EXTRA PROBLEM !** First lets construct the differential equation for the amount $x(t)$ of salt [kg] as a function of time t.

To set up a differential equation for $x(t)$, we estimate the change Δx in x during the time interval $[t,\ t+\Delta t]$. Let the solution with concentration of c_i [gr/lt] flows into tank with rate of r_i [lt/sec] and the mixture (with concentration c_0) flows out at the constant rate r_0 [lt/sec], then

$$\Delta x = \{\text{grams input}\} - \{\text{grams output}\} = r_i c_i \Delta t - r_0 c_0 \Delta t.$$

Note that the unit of the R.H.S. of the above equation is [kg]. Hence

$$\frac{\Delta x}{\Delta t} = r_i c_i - r_0 c_0.$$

If we tale the limit $\Delta t \to 0$ then

$$\frac{dx}{dt} = r_i c_i - r_0 c_0,$$

in which r_i, c_i and r_0 are constants but c_0 denotes the variable concentration $c_0 = x(t)/V(t)$ where $V(t)$ denotes the volume of the solution at any time t. Therefore $x(t)$ satisfies the following first order linear D.E.

$$\frac{dx}{dt} + \frac{r_0}{V}x = r_i c_i.$$

In this problem Volume of the solution is given as $V = 1000$ lt, and initially the amount of the salt is given as 100 kg, i.e. $x(0) = 100$ kg. Moreover, $r_i = r_0 = 5$ lt/sec and $c_i = 0$ kg/lt (since pure water is pumped into tank). Therefore, we have the following I.V.P.

$$x' + \frac{x}{200} = 0, \qquad x(0) = 100kg.$$

Then the solution is

$$x(t) = 100e^{-t/200}.$$

Hence we should the above equation for t when $x = 10$, and we find $t = 461$ sec.

34 (34)) EXTRA PROBLEM ! Let $x(t)$ denote the amount of pollutants in the lake after t days, measured in millions of cubic feet [mft^3]. The volume of the lake is 8000 mft^3, and the initial amount $x(0) = x_0$ of pollutants is $x_0 = (0.25\%)(8000) = 20$ mft^3. We want to know when $x(t) = (0.10\%)(8000) = 8$ mft^3. We set up the differential equation in infinitesimal form by writing

$$dx = [\text{in}] - [\text{out}] = (0.0005)(500)dt - \frac{x}{8000}\, 500\, dt,$$

which simplifies to

$$\frac{dx}{dt} = \frac{1}{4} - \frac{x}{16}.$$

Using the integrating factor $\rho = e^{t/16}$, we obtain the solution

$$x(t) = 4 + 16e^{-t/16}$$

which satisfies the I.C. $x(0) = 20$. Finally, we find that $x = 8$ when $t = 16 \ln 4 \approx 22.2$ days.

37 (37)) EXTRA PROBLEM ! The volume of brine in the tank after t min is $V(t) = 100 + 2t$ gal, so the I.V.P. is

$$\frac{dx}{dt} = 5 - \frac{3x}{100 + 2t}, \qquad x(0) = 50.$$

The integrating factor is $\rho(t) = (100 + 2t)^{3/2}$ leads to the solution

$$x(t) = (100 + t) - \frac{50000}{100 + 2t)^{3/2}},$$

such that $x(0) = 50$. The tank is full after $t = 150$ min, at which time $x(150) = 393.75$ lb.