BİLKENT UNIVERSITY
Department of Mathematics

MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 22

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Homework problems from the 2nd Edition, SECTION 6.3

15(15))[2] The C.E.

$$p(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)^2 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 1,\ \lambda_2 = 2$. The eigenvectors for $\lambda_1 = 1$:

$$\begin{aligned} 3v_{11} - 3v_{12} &= 0 \\ 2v_{11} - 2v_{12} &= 0 \end{aligned} \qquad (1)$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 2$:

$$\begin{aligned} 2v_{21} - 3v_{22} &= 0 \\ 2v_{21} - 3v_{22} &= 0 \end{aligned} \qquad (2)$$

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Therefore,

$$\mathbf{P} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad \mathbf{P}^{-1} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}.$$

$$\mathbf{A}^5 = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 32 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 94 & -93 \\ 62 & -61 \end{bmatrix}.$$

7(7)) The C.E.

$$p(\lambda) = -(\lambda - 1)(\lambda - 2)^2 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 2$. The eigenvectors for $\lambda_1 = 1$:

$$\begin{aligned} 3v_{12} &= 0 \\ v_{12} &= 0 \\ v_{13} &= 0 \end{aligned}$$

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

[2] The number in the parenthesis denotes the problem number in the International Edition of the textbook

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_2 = 2$:

$$\begin{aligned} 3v_{22} - v_{21} &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

Therefore,

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \mathbf{P}^{-1} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$\mathbf{A}^5 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 93 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix}.$$

10(10)) The C.E.

$$p(\lambda) = -(\lambda - 1)(\lambda - 2)^2 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 2$. The eigenvectors for $\lambda_1 = 1$:

$$\begin{aligned} 3v_{11} - 3v_{12} + v_{13} &= 0 \\ 2v_{11} - 2v_{12} + v_{13} &= 0 \\ v_{13} &= 0 \end{aligned}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda_2 = 2$:

$$\begin{aligned} 2v_{21} - 3v_{22} + v_{23} &= 0 \\ 2v_{21} - 3v_{22} + v_{23} &= 0 \\ 0 &= 0 \end{aligned}$$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

Therefore,

$$\mathbf{P} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \mathbf{P}^{-1} = \frac{1}{2} \begin{bmatrix} -4 & 6 & -2 \\ 0 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}.$$

$$\mathbf{A}^5 = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{bmatrix} \frac{1}{2} \begin{bmatrix} -4 & 6 & -2 \\ 0 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 94 & -93 & 31 \\ 62 & -61 & 31 \\ 0 & 0 & 32 \end{bmatrix}.$$

35(35)) The fact that each $|\lambda| = 1$, so $\lambda = \pm 1$, implies that $\mathbf{D}^n = \mathbf{I}$ if n is even, in which case $\mathbf{A}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1} = \mathbf{P}\mathbf{I}^n\mathbf{P}^{-1} = \mathbf{I}$.

36(36)) We find immediately that $\mathbf{A}^2 = \mathbf{I}$ so $\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \mathbf{I}\mathbf{A} = \mathbf{A}, \quad \mathbf{A}^4 = \mathbf{A}^3\mathbf{A} = \mathbf{A}^2 = \mathbf{I}$ and so forth.

37(37)) We find immediately that $\mathbf{A}^2 = -\mathbf{I}$, so $\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = -\mathbf{I}\mathbf{A} = -\mathbf{A}, \quad \mathbf{A}^4 = \mathbf{A}^3\mathbf{A} = -\mathbf{A}^2 = \mathbf{I}$ and so forth.