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MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 21

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Homework problems from the 2nd Edition, SECTION 6.2

15(15))[2] The characteristic eq.

$$p(\lambda) = -\lambda^3 + 2\lambda^2 - \lambda = -\lambda(\lambda - 1)^2 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 1$. The eigenvectors for $\lambda_1 = 0$:

$$\begin{aligned} 3v_{11} - 3v_{12} + v_{13} &= 0 \\ 2v_{11} - 2v_{12} + v_{13} &= 0 \\ v_{13} &= 0 \end{aligned}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda_2 = 1$:

$$\begin{aligned} 2v_{21} - 3v_{22} + v_{23} &= 0 \\ 2v_{21} - 3v_{22} + v_{23} &= 0 \\ 0 &= 0 \end{aligned}$$

The eigenspace of $\lambda_2 = 1$ is 2-dimensional. If we let $v_{22} = 0$ and $v_{23} = 2$ we obtain

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

and if we let $v_{22} = 2$ and $v_{23} = 0$ we obtain

$$\vec{v}_3 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}.$$

Therefore,

$$P = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

[2] The number in the parenthesis denotes the problem number in the International Edition of the textbook

19(19)) The characteristic eq.

$$p(\lambda) = -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = -(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3$. The eigenvectors for $\lambda_1 = 1$:

$$\begin{aligned} v_{12} - v_{13} &= 0 \\ -2v_{11} + 3v_{12} - v_{13} &= 0 \\ -4v_{13} + 4v_{12} &= 0 \end{aligned}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 2$:

$$\begin{aligned} -v_{21} + v_{22} - v_{23} &= 0 \\ -2v_{21} + 2v_{22} - v_{23} &= 0 \\ -4v_{21} + 4v_{22} - v_{23} &= 0 \end{aligned}$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda_3 = 3$:

$$\begin{aligned} -2v_{31} + v_{32} - v_{33} &= 0 \\ -2v_{31} + v_{32} - v_{33} &= 0 \\ -4v_{31} + 4v_{32} - 2v_{33} &= 0 \end{aligned}$$

$$\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

Therefore,

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

21(21)) The characteristic eq.

$$p(\lambda) = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda - 1)^3 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = 1$. The eigenvectors for $\lambda_1 = 1$:

$$\begin{aligned} -v_{11} + v_{12} &= 0 \\ -v_{11} + v_{12} &= 0 \\ -v_{11} + v_{12} &= 0 \end{aligned}$$

The eigenspace of $\lambda_1 = 1$ is 2-dimensional. We obtain the eigenvectors $\vec{v}_1$ by setting $v_{12} = 0,\ v_{22} = 1$ and the eigenvector $\vec{v}_2$ by setting $v_{12} = 1,\ v_{22} = 0$. Since $\mathbf{A}$ has only two L.I. eigenvectors, it is not diagonalizable.

29(29)) If $\mathbf{A}$ is similar to $\mathbf{B}$ and $\mathbf{B}$ is similar to $\mathbf{C}$, so

$$\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}, \qquad \mathbf{B} = \mathbf{Q}^{-1}\mathbf{C}\mathbf{Q}.$$

Then

$$\mathbf{A} = \mathbf{P}^{-1}(\mathbf{Q}^{-1}\mathbf{C}\mathbf{Q})\mathbf{P} = (\mathbf{Q}\mathbf{P})^{-1}\mathbf{C}(\mathbf{Q}\mathbf{P}) = \mathbf{S}^{-1}\mathbf{C}\mathbf{S},$$

where $\mathbf{S} = \mathbf{Q}\mathbf{P}$, so $\mathbf{A}$ is similar to $\mathbf{C}$.

31(31)) If $\mathbf{A}$ is similar to $\mathbf{B}$, $\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}\mathbf{P}$. If we take the inverse of both sides

$$\mathbf{A}^{-1} = (\mathbf{P}^{-1}\mathbf{B}\mathbf{P})^{-1} = \mathbf{P}^{-1}\mathbf{B}^{-1}\mathbf{P}.$$

So, $\mathbf{A}^{-1}$ is similar to $\mathbf{B}^{-1}$.

34(34)) The C.E. of the given 2×2 matrix $\mathbf{A}$ is

$$p(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

The discriminant Δ of this quadratic equation is

$$\Delta = (a+d)^2 - (ad - bc) = (a-d)^2 + 4bc.$$

a) If $\Delta > 0$, then $\mathbf{A}$ has two distinct roots (eigenvalues) and hence two L.I. eigenvectors, therefore it is diagonalizable.
b) If $\Delta < 0$, then $\mathbf{A}$ has no real eigenvalues and hence no real eigenvectors and therefore it is not diagonalizable.
c) If $\Delta = 0$, for both

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

but $\mathbf{A}$ has only the single eigenvalue $\lambda = 1$ and the single eigenvector

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and therefore $\mathbf{A}$ is not diagonalizable.

35(35)) Three eigenvectors associated with three distinct eigenvalues can be arranged in six different orders as the column vectors of the diagonalizing matrix $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}^T$.

36(36)) The fact that the matrices $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues (with the same multiplicities) implies that they are both similar to the same diagonal matrix $\mathbf{D}$ having these eigenvalues as its diagonal elements. But two matrices that are similar to a third matrix are (by Problem # 29) similar to one another.

37(37)) If $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ with $\mathbf{P}$ the eigenvector matrix of $\mathbf{A}$ and $\mathbf{D}$ its diagonal matrix of eigenvalues, then $\mathbf{A}^2 = (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})(\mathbf{P}\mathbf{D}\mathbf{P}^{-1}) = \mathbf{P}\mathbf{D}^2\mathbf{P}^{-1}$. Thus the same (eigenvector) matrix $\mathbf{P}$ diagonalizes $\mathbf{A}^2$, but the resulting diagonal (eigenvalue) matrix $\mathbf{D}^2$ is the square of the one for $\mathbf{A}$. The diagonal elements of $\mathbf{D}^2$ are the eigenvalues of $\mathbf{A}^2$ and the diagonal elements of $\mathbf{D}$ are the eigenvalues of $\mathbf{A}$, so the former are the squares of the latter.

38(38)) If the $n \times n$ matrix $\mathbf{A}$ has n L.I. eigenvectors associated with the single eigenvalue λ, then $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ with $\mathbf{D} = \lambda\mathbf{I}$, so $\mathbf{A} = \mathbf{P}(\lambda\mathbf{I})\mathbf{P}^{-1} = \lambda\mathbf{P}\mathbf{P}^{-1} = \lambda\mathbf{I} = \mathbf{D}$.