BİLKENT UNIVERSITY
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MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 20

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Homework problems from the 2_{nd} Edition, SECTION 6.1

3(3))[2] The characteristic eq.

$$p(\lambda) = \lambda^2 - 7\lambda + 10 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 2,\ \lambda_2 = 5$. The eigenvectors for $\lambda_1 = 2$:

$$\begin{aligned} 6v_{11} - 6v_{12} &= 0 \\ 3v_{11} - 3v_{12} &= 0 \end{aligned}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 5$:

$$\begin{aligned} 3v_{21} - 6v_{22} &= 0 \\ 3v_{21} - 6v_{22} &= 0 \end{aligned}$$

$$\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

6(6)) The characteristic eq.

$$p(\lambda) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 2,\ \lambda_2 = 3$. The eigenvectors for $\lambda_1 = 1$:

$$\begin{aligned} 4v_{11} - 4v_{12} &= 0 \\ 3v_{11} - 3v_{12} &= 0 \end{aligned}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 3$:

$$\begin{aligned} 3v_{21} - 4v_{22} &= 0 \\ 3v_{21} - 4v_{22} &= 0 \end{aligned}$$

$$\vec{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

[2] The number in the parenthesis denotes the problem number in the International Edition of the textbook

14(14)) The characteristic eq.

$$p(\lambda) = -\lambda^3 + 7\lambda^2 - 10\lambda + 6 = -\lambda(\lambda - 2)(\lambda - 5) = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 0,\ \lambda_2 = 2,\ \lambda_3 = 5$. The eigenvectors for $\lambda_1 = 0$:

$$\begin{aligned} 5v_{11} &= 0 \\ 4v_{11} - 4v_{12} - 2v_{13} &= 0 \\ -2v_{11} + 12v_{12} + 6v_{13} &= 0 \end{aligned}$$

$$\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

For $\lambda_2 = 2$:

$$\begin{aligned} 3v_{21} &= 0 \\ 4v_{21} - 6v_{22} - 2v_{23} &= 0 \\ -2v_{21} + 12v_{22} + 4v_{23} &= 0 \end{aligned}$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

For $\lambda_3 = 5$:

$$\begin{aligned} 0 &= 0 \\ 4v_{31} - 9v_{32} - 2v_{33} &= 0 \\ -2v_{31} + 12v_{32} + v_{33} &= 0 \end{aligned}$$

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

21(21)) The characteristic eq.

$$p(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = -(\lambda - 1)(\lambda - 2)^2 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = 1,\ \lambda_2 = \lambda_3 = 2$. The eigenvectors for $\lambda_1 = 1$:

$$\begin{aligned} 3v_{11} - 3v_{12} + v_{13} &= 0 \\ 2v_{11} - 2v_{12} + v_{13} &= 0 \\ v_{13} &= 0 \end{aligned}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda_2 = 2$:

$$\begin{aligned} 2v_{21} - 3v_{21} + v_{23} &= 0 \\ 2v_{21} - 3v_{22} + v_{23} &= 0 \\ 0 &= 0 \end{aligned}$$

$$\vec{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

28(28)) The characteristic eq.

$$p(\lambda) = \lambda^2 + 36 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = -6i,\ \lambda_2 = 6i$. The eigenvectors for $\lambda_1 = -6i$:

$$\begin{aligned} 6iv_{11} - 6v_{12} &= 0 \\ 6v_{11} + 6iv_{12} &= 0 \end{aligned}$$

$$\vec{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

For $\lambda_2 = 6i$:

$$\begin{aligned} -6iv_{21} - 6v_{22} &= 0 \\ 6v_{21} - 6iv_{22} &= 0 \end{aligned}$$

$$\vec{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

31(31)) The characteristic eq.

$$p(\lambda) = \lambda^2 + 144 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = -12i,\ \lambda_2 = 12i$. The eigenvectors for $\lambda_1 = -12i$:

$$\begin{aligned} 12iv_{11} + 24v_{12} &= 0 \\ -6v_{11} + 12iv_{12} &= 0 \end{aligned}$$

$$\vec{v}_1 = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

For $\lambda_2 = 12i$:

$$\begin{aligned} -12iv_{21} + 24v_{22} &= 0 \\ -6v_{21} - 12iv_{22} &= 0 \end{aligned}$$

$$\vec{v}_2 = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

35(35))

a) Since $\mathbf{I}^T = \mathbf{I}$, and $\det \mathbf{A} = \det \mathbf{A}^T$ where $\mathbf{I}$ and $\mathbf{A}$ are $n \times n$ identity matrix and square matrix respectively. Hence,

$$(\mathbf{A} - \lambda \mathbf{I})^T = (\mathbf{A}^T - \lambda \mathbf{I})$$

and then it follows that

$$|\mathbf{A} - \lambda \mathbf{I}| = |\mathbf{A}^T - \lambda \mathbf{I}|.$$

Thus the matrices $\mathbf{A}$ and $\mathbf{A}^T$ have the same characteristic polynomial, and therefore have the same eigenvalues.

b) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

which has the C.E. (Characteristic Polynomial, C.P.) $(\lambda - 1)^2 = 0$, and hence it has the single eigenvalue $\lambda = 1$. Then

$$\mathbf{A} - \mathbf{I} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and it follows that $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the only associated multiple eigenvector. The transpose

$$\mathbf{A}^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

has the same C.E. and eigenvalue, but similarly $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the only eigenvector. Therefore, $\mathbf{A}$ and $\mathbf{A}^T$ has the same eigenvalue but different eigenvectors.

39(39)) If the characteristic equation of the $n \times n$ matrix $\mathbf{A}$ with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ (not necessarily distinct) is written in the factored form

$$(\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n) = 0$$

then it should be clear that upon multiplying out the factors the coefficient of λ^{n-1} will be $-(\lambda_1 + \lambda_2 + ...\lambda_n)$. But according to Problem # 38, this coefficient also equals $\mathrm{tr}\mathbf{A}$. Therefore

$$\lambda_1 + \lambda_2 + ...\lambda_n = \mathrm{tr}\mathbf{A} = a_{11} + a_{22} + ... + a_{nn}.$$

40(40)) Since $\mathrm{tr}\mathbf{A} = 12$, $\det \mathbf{A} = 60$, so the C.P. of $\mathbf{A}$ is $p(\lambda) = -\lambda^3 + 12\lambda^2 + c_1\lambda + 60 = 0$. For $\lambda = 1$

$$p(1) = |\mathbf{a} - \mathbf{I}| = \begin{vmatrix} 31 & -67 & 47 \\ 7 & -15 & 13 \\ -7 & 15 & -7 \end{vmatrix} = 24$$

gives $c_1 = -47$. So the C.E.

$$p(\lambda) = -\lambda^3 + 12\lambda^2 - 47\lambda + 60 = (\lambda - 3)(\lambda - 4)(\lambda - 5) = 0.$$

We obtain the following associated eigenvectors

$$\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 5 \\ 7 \\ 7 \end{bmatrix}, \qquad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix},$$

respectively.