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MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 2

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Homework problems from the 2nd Edition, SECTION 1.4

3 **(3)**)[2] The given D.E. is separable, so if we separate the variables and integrate both sides, we obtain

$$\int \frac{dy}{y} = \int \sin x \, dx + c, \qquad c = \text{constant}$$

Integration yields

$$\ln y = -\cos x + c$$

If we take the exponential of both sides we get the following general solution:

$$y(x) = e^{-\cos x + c}$$

6 **(6)**)

$$\int \frac{dy}{\sqrt{y}} = \int 3\sqrt{x} dx$$

$$2\sqrt{y} = 2x^{3/2} + c, \qquad c = \text{constant}$$

$$y = \phi(x) = (x^{3/2} + C)^2, \qquad C = \text{constant}.$$

9 **(9)**) Similar to previous problem, D.E is separable:

$$\int \frac{dy}{y} = \int \frac{2}{1 - x^2} \, dx$$

If we use the partial fraction decomposition, we obtain

$$\int \frac{dy}{y} = \int \left[\frac{1}{1+x} + \frac{1}{1-x} \right] dx$$

Hence,

$$\ln y = \ln(1 - x) - \ln(1 + x) + \ln c$$

Solving for y gives

$$y(x) = c \frac{1 + x}{1 - x}.$$

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

[2] The number in the parenthesis denotes the problem number in the International Edition

10 (10))

$$\int \frac{dy}{(1+y)^2} = \int \frac{dx}{(1+x)^2}$$

Integration yields,

$$-\frac{1}{1+y} = -\frac{1}{1+x} - c = -\frac{1+c(1+x)}{1+x}, \qquad c = \text{constant}$$

15 (15))

$$\int \left[\frac{2}{y^2} - \frac{1}{y^4}\right] dy = \int \left[\frac{1}{x} - \frac{1}{x^2}\right] dx.$$

Hence the implicit solution is

$$-\frac{2}{y} + \frac{1}{3y^2} = \ln|x| + \frac{1}{x} + c, \qquad c = \text{constant}.$$

18 (18)) Write the given D.E. in form of

$$x^2 y' = 1 - x^2 + y^2 - x^2 y^2 = (1 - x^2)(1 + y^2)$$

then it is separable and hence

$$\int \frac{dy}{1+y^2} = \int \left[\frac{1}{x^2} - 1\right] dx$$

$$\tan^{-1} y = -\frac{1}{x} - x + c, \qquad c = \text{constant}.$$

Therefore

$$y(x) = \tan\left[c - \frac{1}{x} - x\right].$$

20 (20))

$$\int \frac{dy}{1+y^2} = \int 3x^2\, dx,$$

$$\tan^{-1} y = x^3 + c, \qquad c = \text{constant}.$$

The solution of the D.E. is

$$y(x) = \tan(x^3 + c)$$

I.C. implies that

$$c = \tan^{-1} 1 = \frac{\pi}{4}.$$

Hence the solution of the I.V.P. is

$$y(x) = \tan(x^3 + (\pi/4)).$$

21 (21))

$$\int 2y\, dy = \int \frac{x}{\sqrt{x^2 - 16}}\, dx, \quad y^2 = \sqrt{x^2 - 16} + c, \qquad c = \text{constant}.$$

I.C. $y(5) = 2$ implies that $c = 1$ therefore the solution of the I.V.P.

$$y^2 = 1 + \sqrt{x^2 - 16}.$$

25 (25))

$$\int \frac{1}{y}\,dy = \int \left[\frac{1}{x} + 2x\right] dx, \qquad \ln y = \ln x + x^2 + \ln c, \qquad c = \text{constant}.$$

The general solution of the D.E. is $y(x) = cxe^{x^2}$. I.C. $y(1) = 1$ implies that $c = e^{-1}$ so the solution of the I.V.P.

$$y(x) = xe^{x^2-1}$$

27 (27))

$$\int e^y dy = \int 6e^{2x} dx, \qquad e^y = 3e^{2x} + c, \qquad y = \phi(x) = \ln(3e^{2x} + c).$$

where c is an integration constant. Initial condition $y(0) = 0$ implies $c = -2$, therefore the solution of the initial value problem is $y = \phi(x) = \ln(3e^{2x} - 2)$.

28 (28))

$$\int \sec^2 y\,dy = \int \frac{dx}{2\sqrt{x}}, \qquad \tan y = \sqrt{x} + c, \qquad c = \text{constant}.$$

The general solution of the D.E. is

$$y(x) = \tan^{-1}(\sqrt{x} + c).$$

I.C. $y(4) = \pi/4$ implies that $c = -1$ so the solution of the I.V.P.

$$y(x) = \tan^{-1}(\sqrt{x} - 1)$$

30 (..)) When we take square roots on both sides of the DE and separate variables, we get

$$\int \frac{dy}{2\sqrt{y}} = \int dx, \qquad \sqrt{y} = x = c, \qquad y = \phi(x) = (x - c)^2$$

where c is an integration constant. This general solution provides the parabolas illustrated in Fig. 1.4.5 in the textbook. Observe that $y = \phi(x)$ is always nonnegative, consistent with both the square root and the original differential equation. We spot also the singular solution $y = \phi(x) = 0$ that corresponds to no value of the constant c.

a. Looking at Fig. 1.4.5, we see immediately that the differential equation $(y')^2 = 4y$ has no solution curve through the point (a, b) if $b < 0$.

b. But if $b \geq 0$ we obviously can combine branches of parabolas with segments along the x-axis to form infinitely many solution curves through (a, b).

c. Finally, if $b > 0$ then on a interval containing (a, b) there are exactly two solution curves through this point, corresponding to the two indicated parabolas through (a, b), one ascending and one descending from left to right.

31 (..)) The formal separation of variables process is the same as the previous problem (Problem 30) where, indeed, we started by taking square roots in $(y')^2 = 4y$ to get the differential equation $y' = 2\sqrt{y}$. But whereas y' can be either positive or negative in the original equation, the latter

equation requires that y' be *nonnegative*. This means that only the *right half* of each parabola $y = (x - c)^2$ qualifies as a solution curve. From the solution curves, we therefore see that through the point (a, b) there passes **a.** No solution curve if $b < 0$,
b. A unique solution curve if $b > 0$,
c. Infinitely many solution curves if $b = 0$, because in this case we can pick any $c > a$ and define the solution $y = \phi(x) = 0$ if $x \leq c$, $y = \phi(x) = (x - c)^2$ if $x \geq c$.

35 (31)) Consider a material that contains $N(t)$ atoms of a certain radioactive isotope at time t. Constant fraction of those radioactive atoms will spontaneously decay (into atoms of another element or into another isotope of the same element) during each unit of time. Therefore, the rate of change of $N(t)$ w.r.t. t must be proportional with $N(t)$ with the proportionality constant $k > 0$. i.e.

$$\frac{dN}{dt} = -kN$$

We have $-$ sign because number of atoms is decreasing and k depends on the particular radioactive isotope. For ^{14}C, $k \approx 0.0001216$ if t is measured in years. If we solve the above D.E., then we get

$$N(t) = N_0 e^{-0.0001216t}.$$

Hence we only need to solve the following equation

$$\frac{N_0}{6} = N_0 e^{-0.0001216t}.$$

for t. We find $t = (\ln 6)/0.0001216 \approx 14735$ years to find the age of the skull.

36 (32)) As in the above problem, the number of ^{14}C atoms after t years is given by $N(t) = 5.0 \times 10^{10} e^{-0.0001216t}$. Hence we only need to solve the equation

$$4.6 \times 10^{10} = 5.0 \times 10^{10} e^{-0.0001216t}$$

for the age

$$t = \frac{\ln(0.5/4.6)}{0.0001216} \approx 686 \text{ years of the relic}$$

Thus it appears not to be a genuine relic of the time of Christ 2000 years ago.

37 (33)) Let $A(t)$ be the number of dollars in a saving account at time t (years), and suppose that the interest is compounded continuously at an annual rate r (for 10% interest is $r = 0.10$). Continuous compounding means that during a short time interval Δt, the amount of interest added to the account is approximately $\Delta A = rA(t)\Delta t$, so that

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} = \frac{\Delta A}{\Delta t} = rA.$$

Therefore we have the following D.E.

$$\frac{dA}{dt} = rA.$$

In this problem $r = 0.08$ and we have the I.C. $A(0) = \$5.000$ and the solution of the I.V.P is

$$A(t) = 5000 e^{0.08t}.$$

Hence the amount in the account after 18 years is given by

$$A(18) = 5000e^{0.08\times 18} \approx 21,103.48 \text{ USD}.$$

40 (36)) Since we have the decaying problem, in this problem we have the same D.E. as in the problem # 35. The **half-life** τ of a radioactive isotope is the time required for half of it to decay. i.e. when $t = \tau$, $N = N_0/2$. Since the solution of the D.E.

$$\frac{dN}{dt} = -kN$$

is

$$N(t) = N_0 e^{kt}$$

so that

$$\frac{N_0}{2} = N_0 e^{k\tau}$$

When we solve for τ, we find that

$$\tau = \frac{\ln 2}{k}$$

In this problem $\tau = 5.27$, thus $k = (\ln 2)/5.27 \approx 0.13153$. Thus the amount of radioactive cobalt left after t years is given by $N(t) = N_0 e^{-0.13153t}$. We therefore solve the equation $N(t) = N_0 e^{-0.13153t} = 0.01N_0$ for $t = (\ln 100)/0.13153 \approx 35.01$ and find that it will be about 35 years unit the region is again inhabitable.