BİLKENT UNIVERSITY
Department of Mathematics

MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS, Solution of Homework set[1] # 19

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1)

a) Corresponding homogenous equation is

$$y'' - y' - 2y = 0$$

Since, it is a constant coefficient D.E. we look for the solution of the form $y = e^{rx}$, then its C.E. is $r^2 - r - 2 = 0$ with the roots $r_1 = 2$ and $r_2 = -1$. Hence the homogenous solution is

$$y_h(x) = c_1 e^{2x} + c_2 e^{-x}, \qquad c_1, c_2 = \text{constant}.$$

Assume

$$y_p = c_1(x)e^{2x} + c_2(x)e^{-x}$$

then

$$y_p' = [2c_1(x)e^{2x} - c_2(x)e^{-x}] + [c_1'(x)e^{2x} + c_2'(x)e^{-x}].$$

We set

$$c_1'(x)e^{2x} + c_2'(x)e^{-x}.$$

Computing y_p'' and substituting in the D.E. gives

$$2c_1'(x)e^{2x} - c_2'(x)e^{-x} = 2e^{-x}.$$

Thus we have two algebraic equations for $c_1'(x)$ and $c_2'(x)$ with the solution

$$c_1'(x) = (2/3)e^{-3x}, \qquad c_2'(x) = -2/3$$

Hence $c_1(x) = -(2/9)e^{-3x}$ and $c_1(x) = -(2x/3)$. Substituting $c_1(x)$ and $c_2(x)$ in y_p gives

$$y_p(x) = -(2x/3)e^{-x}.$$

b) Similarly, first find the homogenous solution

$$y_h(x) = c_1 e^{-x} + c_2 x e^{-x} \qquad c_1, c_2 = \text{constant}.$$

Assume

$$y_p = c_1(x)e^{-x} + c_2(x)xe^{-x}.$$

$c_1'(x)$ and $c_2'(x)$ satisfy

$$c_1'(x)e^{-x} + c_2'(x)xe^{-x} = 0, \quad \text{and} \quad -c_1'(x)e^{-x} + c_2'(x)[e^{-x} - xe^{-x}] = 3e^{-x}$$

with the solutions

$$c_1'(x) = -3x, \quad \text{and} \quad c_2'(x) = 3.$$

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

Therefore $y_p(x) = (3/2)x^2e^{-x}$.

2)
a) Since $\cos x$ and $\sin x$ are homogenous solutions, we assume

$$y_p = c_1(x)\cos x + c_2(x)x\sin x.$$

The system of equations for $c_1'(x)$ and $c_2'(x)$ is

$$c_1'\cos x + c_2'\sin x = 0, \quad \text{and} \quad -c_1'\sin x + c_2'\cos x = \tan x$$

With the solutions

$$c_1' = -\sin^2 x/\cos x = -\sec x + \cos x, \quad \text{and} \quad -c_2' = -\cos x.$$

Thus

$$c_1(x) = \sin x - \ln(\tan x + \sec x), \quad \text{and} \quad c_2(x) = -\cos x.$$

Hence

$$y_p(x) = c_1\cos x + c_2\sin x - (\cos x)\ln(\tan x + \sec x), \qquad c_1, c_2 = \text{constant}.$$

b) Corresponding homogenous eq. is

$$y'' + 4y' + 4y = 0.$$

which is a constant coefficient linear D.E. and its C.E is

$$r^2 + 4r + 4 = 0$$

with the roots $r = -2$ with multiplicity $k = 2$. Therefore, the homogenous solution

$$y_h(x) = c_1e^{-2x} + c_2xe^{-2x}. \qquad c_1, c_2 = \text{constant}.$$

Since the R.H.S. is not a solution of a constant coefficient linear differential equation, we should use the variation of parameters to find y_p. Let

$$y_p = c_1(x)e^{-2x} + c_2(x)xe^{-2x}.$$

$c_1(x)$ and $c_2(x)$ are given by

$$c_1 = \int -\frac{y_2(x)R(x)}{W(y_1, y_2)(x)}dx, \quad \text{and} \quad c_2 = \int \frac{y_1(x)R(x)}{W(y_1, y_2)(x)}dx.$$

where $R(x) = x^{-2}e^{-2x}$ and the Wronskian $W(y_1, y_2)(x) = e^{-4x}$. Therefore

$$c_1(x) = -\ln x, \quad \text{and} \quad c_2(x) = -\frac{1}{x}.$$

Particular solution is

$$y_p = -e^{-2x}\ln x - e^{-2x}.$$

and the general solution is

$$y(x) = c_1e^{-2x} + c_2xe^{-2x} - e^{-2x}\ln x. \qquad c_1, c_2 = \text{constant}.$$

c) Two L.I. solutions of the homogenous equation are $y_1(x) = e^{3x}$ and $y_2(x) = e^{2x}$ and the Wronskian $W(y_1, y_2)(x) = -e^{5x}$. Therfore,

$$y_p(x) = e^{3x} \int \frac{e^{2t} R(t)}{e^{5t}} dt - e^{2x} \int \frac{e^{3t} R(t)}{e^{5t}} dt,$$

$$y_p(x) = e^{3x} \int [e^{3(x-t)} - e^{2(x-t)}] R(t) dt.$$

3)

a) That x and xe^x are solutions of the homogenous D.E. can be verified by direct substitution. Thus we assume

$$y_p = c_1(x)x + c_2(x)xe^x.$$

Following the pattern of earlier problems we find

$$c_1' + c_2' xe^x = 0, \quad \text{and} \quad -c_1' + (x+1)e^x c_2' = 2x.$$

Note that the R.D.S. of the D.E. is $R(x) = 2x$. The solution of these equations gives $c_1'(x) = -2$ and $c_2'(x) = 2e^{-x}$. Hence

$$c_1(x) = -2x \quad \text{and} \quad c_2(x) = -2e^{-x}.$$

and

$$y_p(x) = -2x^2 - 2x.$$

However, since x is a solution of the homogenous D.E. we can choose as the particular solution

$$y_p(x) = -2x^2.$$

b) Since $y_1 = x^{-1/2} \sin x$, $\quad y_2 = x^{-1/2} \cos x$, then the Wronskian is

$$W(y_1, y_2)(x) = -\frac{1}{x}.$$

If we put the D.E. in normal form, then $R(x) = 3x^{-1/2} \sin x$ and thus

$$c_1'(x) = -\frac{y_1(x)R(x)}{W(y_1, y_2)} = 3 \sin x \cos x$$

and

$$c_2'(x) = \frac{y_2(x)R(x)}{W(y_1, y_2)} = -3 \sin^2 x = \frac{3}{2}(-1 + \cos 2x).$$

Hence

$$c_1(x) = \frac{3}{2} \sin^2 x, \quad \text{and} \quad c_2(x) = -\frac{3x}{2} + \frac{3}{4} \sin 3x.$$

Therefore,

$$y_p(x) = \frac{3 \sin^2 x}{2} \frac{\sin x}{\sqrt{x}} + \left(-\frac{3x}{2} + \frac{3 \sin 2x}{4} \right) \frac{\cos x}{\sqrt{x}},$$

$$y_p(x) = \frac{3 \sin^2 x}{2} \frac{\sin x}{\sqrt{x}} + \left(-\frac{3x}{2} + \frac{3 \sin x \cos x}{4} \right) \frac{\cos x}{\sqrt{x}},$$

$$y_p(x) = \frac{3 \sin 2x}{2\sqrt{x}} - \frac{3\sqrt{x} \cos x}{2}.$$

4)

a) Homogenous solution is $y_h(x) = c_1 + c_2 \cos x + c_3 \sin x$ where c_1, c_2, c_3 are arbitrary constants, and thus we assume a particular solution of the form

$$y_p(x) = c_1(x) + c_2(x) \cos x + c_3(x) \sin x.$$

Differentiating y_p and letting the sum of the terms containing the derivative of c_1, c_2 and c_3 zero, we obtain

$$y_p' = -c_2 \sin x + c_3 \cos x, \quad \text{and} \quad c_1' + c_2' \cos x + c_3' \sin x = 0.$$

Continuing this process we obtain

$$y_p'' = -c_2 \cos x - c_3 \sin x, \qquad y_p''' = c_2 \sin x - c_3 \cos x - c_2' \cos x - c_3' \sin x$$

and

$$-c_2' \sin x + c_3' \cos x = 0.$$

Substituting y_p and its derivatives into D.E. we obtain the third equation for the derivatives of c_1, c_2 and c_3:

$$-c_2' \cos x - c_3' \sin x = \tan x.$$

If we solve c_1', c_2' and c_3' from above equations we get

$$c_1'(x) = \tan x, \qquad c_2'(x) = -\sin x, \qquad c_3'(x) = -\frac{\sin^2 x}{\cos x}.$$

If we integrate and c_1', c_2' and c_3' and substitute in y_p, we get

$$y_p(x) = -\ln \cos x - (\sin x) \ln(\sec x + \tan x).$$

b) Since the homogenous solutions are $y_1 = e^x,\ y_2 = \cos x,\ y_3 = sinx$ a particular solution is of the form

$$y_p(x) = c_1(x)e^x + c_2(x) \cos x + c_3(x) \sin x.$$

Differentiating y_p and making the same assumptions, we obtain

$$c_1'(x)e^x + c_2'(x) \cos x + c_3'(x) \sin x = 0, \qquad c_1'(x)e^x - c_2'(x) \sin x + c_3'(x) \cos x = 0,$$

and

$$c_1'(x)e^x - c_2'(x) \cos x - c_3'(x) \sin x = R(x).$$

Solving these equations, integrating and substituting into y_p yield

$$y_p(x) = \frac{1}{2}\left[e^x \int e^{-x} R(x)dx + \cos x \int (\sin x - \cos x)R(x)dx - \sin x \int (\sin x + \cos x)r(x)dx\right].$$

c) Homogenous solutions are $y_1 = e^x,\ y_2 = xe^x$, and $y_3 = x^2 e^x$. So

$$y_p(x) = c_1(x)e^x + c_2(x)xe^x + c_3 x^2 e^x.$$

The equations for c_1', c_2', c_3' are

$$c_1'(x) = \frac{1}{2}x^2 e^{-x} R(x), \quad c_2'(x) = -xe^{-x}R(x), \quad c_3(x) = \frac{1}{2}e^{-x}R(x).$$

Integrating and substituting into y_p yields

$$y_p(x) = -xe^x \ln |x|.$$