BİLKENT UNIVERSITY
Department of Mathematics

MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 17

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1)
a) Substituting $y = e^{rx}$ into the D.E., we get the C.E. $r^2 - 2r + 1 = 0$ which gives $r_1 = r_2 = 1$. Then two L.I. solutions $y_1 = e^x$ and $y_2 = xe^x$. So the general solution is $y(x) = c_1e^x + c_2xe^x$.
Or, we can use the method of reduction of order if $y_1 = e^x$, then the second L.I. solution is of the form $y_2 = y_1v(x) = e^xv(x)$. D.E yields $v'' = 0$. Solving the D.E. for v gives $v(x) = c_1 + c_2x, \quad c_1, c_2 =$ constant. The general solution of the given D.E. can be obtained from $y = e^xv(x)$.
b) Substituting $y = e^{rx}$ into the D.E. we get the C.E. $25r^2 - 20r + 4 = 0$ which may be written as $(5r - 2)^2 = 0$. The roots of the C.E. are $r_1 = r_2 = 2/5$.
Thus, $y(x) = c_1e^{2x/5} + c_2xe^{2x/5}, \quad c_1, c_2 =$ constant.
c) Substituting $y = e^{rx}$ into the D.E. we get the C.E. $r^2 - 2r + 10 = 0$ and hence the roots are $r_1 = 1 + 3i, \quad r_2 = 1 - 3i$. Thus, $y(x) = c_1e^x \cos 3x + c_2e^x \sin 3x, \quad c_1, c_2 =$ constant.

2)
a) The C.E. is $r^2 + 4r + 4 = (r + 2)^2 = 0$, which has the repeated root $r_1 = r_2 = r = -2$. Then the general solution is $y(x) = c_1e^{-2x} + c_2xe^{-2x}, \quad c_1, c_2 =$ constant. Since the I.C. are given at $x = -1$, we can write the general solution as $y(x) = c_1e^{-2(x+1)} + c_2xe^{-2(x+1)}$. Then $y'(x) = -2c_1e^{-2(x+1)} + 2c_2e^{-2(x+1)} - 2c_2xe^{-2(x+1)}$ and hence $c_1 - c_2 = 2$ and $-c_1 + 3c_2 = 1$ which yields $c_1 = 7$ and $c_2 = 5$. Thus the general solution is $y(x) = 7e^{-2x} + 5xe^{-2x}$, and decaying exponentially as $x \to \infty$.
b) C.E. is $9r^2 - 12r + 4 = 0$ and its roots are $r_1 = r_2 = 2/3$. Therefore, the general solution of the D.E. is

$$y = c_1e^{2x/3} + c_2xe^{2x/3}, \qquad c_1, c_2 = \text{constant}$$

The initial conditions yield $c_1 = 2$ and $c_2 = -7/3$. Since $r > 0$, $y \to \infty$ as $x \to \infty$.
c) C.E. is $9r^2 + 6r + 82 = 0$ and its roots are $r_{1,2} = -(1/3) \pm 3i$. Therefore, the general solution of the D.E. is

$$y = c_1e^{-x/3} \cos 3x + c_2e^{-x/3} \sin 3x, \qquad c_1, c_2 = \text{constant}$$

The initial conditions yield $c_1 = -1$ and $c_2 = 5/9$. Since the real part of the roots are negative, $y \to 0$ as $x \to \infty$.

3)
a) Let $y_2 = xv(x)$. Then $y_2' = v + xv', \ y_2'' = xv'' + 2v'$ and substituting in the D.E., we obtain

$$xv'' + 4v' = 0, \qquad x > 0$$

If we let $w = v'$ then we have

$$xw' + 4w = 0, \qquad x > 0$$

The general solution of the above equation is $w(x) = c_1x^{-4}, \quad c_1 =$ constant. Since $v' = w$ integration of $w(x)$ yields $v(x) = k_1x^{-3} + k_2, \quad k_1, k_2 =$ constant. Multiplying v with x gives $y(x) = k_1x^{-2} + k_2x$.

[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

Therefore the second L.I. solution is $y_2 = x^{-2}$.
b) Let $y_2 = y_1 v(x)$. Substituting this form of y_2 in the D.E. gives

$$x(y_1 v)'' - (y_1 v)' + 4x^3 (y_1 v) = 0.$$

On carrying out the differentiations and making use of the fact that y_1 is a solution , one obtains

$$x y_1 v'' + (2x y_1' - y_1) v' = 0.$$

This is a first order linear equation for v', which has a solution

$$v' = c\frac{x}{(\sin x^2)^2}, \qquad c = \text{constant}.$$

Setting $u = x^2$ allows integration of this to get

$$v = c_1 \cot x^2 + c_2, \qquad c_1, c_2 = \text{constant}.$$

Setting $c_1 = 1$ and $c_2 = 0$ and multiplying by $y_1 = \sin x^2$, we obtain $y_2(x) = \cos x^2$.
c) Let $y_2 = e^x v(x)$. Then $y_2' = e^x(v + v')$, $y_2'' = e^x(v'' + 2v' + v)$ and substituting in the D.E., we obtain the following first order linear D.E. for v':

$$(x-1)v'' + (x-2)v' = 0, \qquad x > 1$$

The above equation has the solution

$$v' = c_1 (x-1) e^{-(x-1)}, \quad c_1 = \text{constant}.$$

Setting $u = x - 1$ and integrating v' yield

$$v(x) = c_1[-(x-1)e^{-(x-1)} - e^{-(x-1)}] + c_2, \qquad c_2 = \text{constant}.$$

Hence the second L.I. solution is $y_2(x) = x$.

4)

$$\left(\frac{y_2}{y_1}\right)' = \frac{y_1 y_2' - y_1' y_2}{y_1^2} = \frac{W(y_1, y_2)}{y_1^2}$$

Use

$$W(y_1, y_2)(x) = W_0 \exp\left[-\int p(x)dx\right], \qquad W_0 = \text{constant}$$

in above equation, then

$$\left(\frac{y_2}{y_1}\right)' = \frac{1}{y_1^2} W_0 \exp\left[-\int p(x)dx\right].$$

Integrating and setting $c = 1$ (since the solution y_2 can be multiplied by a constant) and taking the constant of integration to be zero, we obtain

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2} dx.$$

a) $p(x) = 3/x$, and $y_1 = 1/x$, and hance

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2} dx = \frac{1}{x} \int \frac{dx}{x} = \frac{1}{x} \ln x.$$

b) $p(x) = 1/x$, and $y_1 = x^{-1/2}\sin x$, and hance

$$y_2(x) = y_1(x)\int \frac{e^{-\int p(x)dx}}{y_1^2}dx = x^{-1/2}\sin x \int \frac{dx}{\sin^2 x} = x^{-1/2}\cos x.$$

5)
a) $f(D)x^n = a_n x^n + na_{n-1}x^{n-1} + n(n-1)a_{n-2}x^{n-2} + n(n-1)(n-2)a_{n-3}x^{n-3} + ... + a_0 n!$
b) $f(D)e^{mx} = f(m)e^{mx}$.
c) The C.E. is $r^4 - 5r^2 + 4 = (r^2-4)(r^2-1) = 0$. Solving for r, we obtain the four solutions e^x, e^{-x}, e^{2x}, e^{-2x}. Since,

$$W(y_1, y_2, .y_3, y_4)(x) = \begin{vmatrix} e^x & e^{-x} & e^{2x} & e^{-2x} \\ e^x & -e^{-x} & 2e^{2x} & -2e^{-2x} \\ e^x & e^{-x} & 4e^{2x} & 4e^{-2x} \\ e^x & -e^{-x} & 8e^{2x} & -8e^{-2x} \end{vmatrix} \neq 0 \qquad (1)$$

for all x, the four functions form a fundamental set of solutions.

6)
a)

$$W' = \frac{d}{dx}\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \frac{d}{dx}[y_1(y_2'y_3'' - y_2''y_3') - y_2(y_1'y_3'' - y_3'y_1'') + y_3(y_1'y_2'' - y_2'y_1'')] \qquad (2)$$

$$W' = y_1(y_2'y_3''' - y_2'''y_3') - y_2(y_1'y_3''' - y_3'y_1''') + y_3(y_1'y_2''' - y_2'y_1''') = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} \qquad (3)$$

Or, use the fact that the derivative of a 3-by-3 determinant is the sum of three 3-by-3 determinants obtained by differentiating the first, second, and third rows respectively.
b) Follow the steps given in the problem.

7) This is the generalization of the reduction of order to the third order equation. Just follow the steps given the question.
a) Let $y = e^x v(x)$. Differentiating four times and substituting into the D.E. yields

$$(2-x)e^x v''' + (3-x)e^x v'' = 0$$

Dividing by $(2-x)e^x$ and letting $w = v''$ we obtain the first order separable equation

$$w' = -\frac{x-3}{x-2}w = \left(-1 + \frac{1}{x-2}\right)w.$$

Separating x and w, integrating, and then solving for w yields

$$w = v'' = c_1(x-2)e^{-x}, \qquad c_1 = \text{constant}.$$

Integrating this twice then gives

$$v = c_1 x e^{-x} + c_2 x + c_3$$

so that

$$y = ve^x = c_1 x + c_2 x e^x + c_3 e^x, \qquad c_1, c_2, c_3 = \text{constant},$$

which is the complete solution, since it contains the given solution $y_1 = e^x$ and three integration constants.

8)
a) Look for the solution of the form $y = e^{rx}$. Substituting in the D.E. gives the following C.E.

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0.$$

Since we have multiple root with multiplicity $k = 2$, the general solution is

$$y(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^x, \qquad c_1, c_2, c_3 = \text{constant},$$

b) C.E. is $r^6 + 1 = 0$. The six roots of -1 are obtained by setting

$$r = (-1)^{1/6} = e^{i(\pi + 2k\pi)/6}, \quad k = 0, 1, 2, 3, 4, 5.$$

They are

$$r_1 = e^{i\pi/6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}, \quad r_2 = i, \quad r_3 = e^{5i\pi/6} = \frac{-\sqrt{3}}{2} + i\frac{1}{2}$$
$$r_4 = e^{7i\pi/6} = \frac{-\sqrt{3}}{2} - i\frac{1}{2}, \quad r_5 = -i, \quad r_6 = e^{11i\pi/6} = \frac{\sqrt{3}}{2} - i\frac{1}{2}.$$

Note that we have three pairs of conjugate roots. The general solution is

$$y(x) = e^{\sqrt{3}x/2}[c_1 \cos(x/2) + c_2 \sin(x/2)] + e^{-\sqrt{3}x/2}[c_3 \cos(x/2) + c_4 \sin(x/2)] + c_5 \cos x + c_6 \sin x,$$

where $c_1, ...c_6 = $ constant.
c) C.E. is $r^6 - r^2 = 0$. The roots are $r = 0$ with multiplicity $k = 2$, and $r^4 - 1 = 0$. Hence $r = e^{i\pi k/2}$, $k = 0, 1, 2, 3$. Therefore, $r_3 = 1$, $r_4 = i$, $r_5 = -1$, $r_6 = -i$. The general solution is

$$y(x) = c_1 + c_2 x + c_3 e^x + c_4 e^{-x} + c_5 \sin x + c_6 \cos x.$$

d) C.E. is $r^4 - 8r = r(r^3 - 8) = 0$. Therefore, $r_1 = 0$ and $r^3 = 8 = 8e^{2i\pi k}, k = $ integer. Hence the roots are

$$r = 2e^{2i\pi k/3}, \quad k = 0, 1, 2$$
$$r_2 = 2, \quad r_3 = 2e^{2i\pi/3}, \quad r_4 = 2e^{4i\pi/3}$$

The general solution is

$$y(x) = c_1 + c_2 e^{2x} + e^{-x}(c_3 \cos \sqrt{3}x + c_4 \sin \sqrt{3}x).$$

e) C.E. is $r^6 + 8 = 0$, so $r^6 = -8 = 8e^{i(\pi + 2k\pi)}, \quad k = $ integer. The roots are

$$r = 8^{1/6} e^{i(\pi + 2k\pi)/6}, \quad k = 0, 1, 2, 3, 4, 5$$

Note that, we have three pairs of conjugate roots.

$$r_1 = 8^{1/6} e^{i\pi/6}, \quad r_2 = 8^{1/6} e^{3i\pi/6}, \quad r_3 = 8^{1/6} e^{5i\pi/6}, \quad r_4 = 8^{1/6} e^{7i\pi/6}, \quad r_5 = 8^{1/6} e^{9i\pi/6}, \quad r_6 = 8^{1/6} e^{11i\pi/6}.$$

Note that we have three pairs of conjugate roots. $\bar{r}_1 = r_6$, $\bar{r}_2 = r_5$, $\bar{r}_3 = r_4$. By using the Euler's formula, we can write the real and imaginary parts of the roots $r_j = \alpha_j \pm i\beta_j$, $j = 1, 2, 3$. Then for each pair, the solution is $y_j = e^{\alpha_j x}(c_j \cos \beta_j x + c_j \sin \beta_j x)$.

9)
a) C.E. is $r^3 + r = 0$ and hence $r = 0, \ i, \ -i$ are the roots and the general solution is

$$y(x) = c_1 + c_2 \cos x + c_3 \sin x, \qquad c_1, c_2, c_3 = \text{constant}.$$

$y(0) = 0$ implies $c_1 + c_2 = 0$, $y'(0) = 1$ implies $c_3 = 1$ and $y''(0) = 2$ implies $c_2 = -2$. Use this last equation in the first to find $c_1 = 2$ and thus

$$y(x) = 2 - 2\cos x + \sin x.$$

b) The general solution is

$$y(x) = c_1 + c_2 x + c_3 e^{2x} + c_4 x e^{2x}$$

Initial conditions imply that $c_1 = -3, \quad c_2 = 2, \quad c_3 = c_4 = 0.$