BİLKENT UNIVERSITY
Department of Mathematics

MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 15

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Homework problems from the 2^{nd} Edition, SECTION 4.5

5)

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Row basis: The three row vectors of **E**.
Column basis: The first, second, and fourth column vectors of **A**.

12) Reduced row echelon form of the matrix is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Row basis: The first three row vectors of **E**.
Column basis: The first, second, and fifth column vectors of **A**.

14) Consider the matrix **A** having the given vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ as its column vectors. The reduced row echelon form **E** of **A** is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 1/5 & 2 \\ 0 & 1 & 2/5 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore the vectors $\vec{v}_1$ and $\vec{v}_2$ are L.I.

16) Similar to the previous problem, the reduced row echelon form of **A** is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, $\vec{v}_1, \vec{v}_2, \vec{v}_4$ and $\vec{v}_5$ are L.I.

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

17) Let the matrix $\mathbf{A} = [\vec{v}_1 \ \ ... \ \ \vec{v}_k \ \ \vec{e}_1 \ \ ... \ \ \vec{e}_n]$. Then its reduced row echelon form $\mathbf{E}$ is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & -3 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Therefore, $\vec{v}_1, \vec{v}_2$ and $\vec{e}_2$ are the basis vectors.

20) Similar to the previous problem consider the matrix $\mathbf{A}$, the its reduced row echelon form of $\mathbf{A}$ is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & -5/2 & 0 & 2 \\ 0 & 1 & 0 & 3/2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

So, $\vec{v}_1, \vec{v}_2, \vec{e}_1$ and $\vec{e}_2$ are the basis vectors.

23) Consider the transpose of the coefficient matrix $\mathbf{A}$. The reduced row echelon form $\mathbf{E}$ of $\mathbf{A}^{\mathbf{T}}$ is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So, the fourth equation is redundant, we have actually three equations.

29) The rank of the $m \times n$ matrix $\mathbf{A}$ is at most $m < n$, and therefore is less than the number n of its column vectors. Hence the column vectors $\vec{a}_1, \ \vec{a}_2, ...\vec{a}_n$, of $\mathbf{A}$ are linearly dependent, so there exists a linear combination

$$y_1\vec{a}_1 + y_2\vec{a}_2 + ... + y_n\vec{a}_n = \vec{0}, \qquad \mathbf{A}\vec{y} = \vec{0}$$

with not all the coefficients being zero. If

$$\vec{x} = [x_1, \ x_2, ...x_n]^T,$$

is one solution of the equation $\mathbf{A}\vec{x} = \vec{b}$ then

$$\mathbf{A}(\vec{x} + \vec{y}) = \mathbf{A}\vec{x} + \mathbf{A}\vec{y} = \vec{b} + \vec{0} = \vec{b},$$

so $\vec{x} + \vec{y}$ is a second different solution. Thus solutions of the equation are not unique.

34) If no row interchanges are involved, then (for any k) the space spanned by the first k row vectors of $\mathbf{A}$ is never changed in the process of reducing $\mathbf{A}$ to the echelon matrix $\mathbf{E}$; this follows immediately from the proof of Theorem 2 in this section. Hence the first r row vectors of $\mathbf{A}$ span the r-dimensional space Row($\mathbf{A}$), and therefore are linearly independent.