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MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 15-A

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Homework problems from the 2nd Edition, SECTION 4.6

4) Yes, the three vectors are mutually orthogonal, since

$$\vec{v}_1.\vec{v}_2 = 3+4+9-12-4 = 0, \quad \vec{v}_1.\vec{v}_3 = 6+4+-12-2+4 = 0, \quad \vec{v}_2.\vec{v}_3 = 18+4-12+6-16 = 0.$$

7) Let $\vec{u} = CB$, $\vec{v} = CA$ and $\vec{w} = AB$, then $a = |\vec{u}|$, $b = |\vec{v}|$ and $c = |\vec{w}|$ so as to verify that $a^2 + b^2 = c^2$. For given vectors $a^2 = 19$, $b^2 = 25$ and $c^2 = 44$.

17) Denote by $\mathbf{A}$ the matrix having the given vectors as its row vectors, and by $\mathbf{E}$ the reduced echelon form of $\mathbf{A}$. From $\mathbf{E}$ we find the general solution of the homogeneous system $\mathbf{A}\tilde{\mathbf{x}} = \tilde{\mathbf{0}}$ in terms of parameters $s, t, ...$ We then get basis vectors $\vec{u}_1, \vec{u}_2, ...$ for the orthogonal complement $V^\perp$ by setting each parameter in turn equal to 1 (and the others then equal to 0). The echelon form $\mathbf{E}$ of the given matrix $\mathbf{A}$ is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & -7 & 19 \\ 0 & 1 & 3 & 5 \end{bmatrix}.$$

$$x_1 = s, \quad x_4 = t, \quad x_2 = -3s + 5t, \quad x_1 = 7s - 19t.$$

Therefore

$$\vec{u}_1 = (7, -3, 1, 10), \quad \vec{u}_2 = (-19, 5, 0, 1).$$

19) Similar to the previous problem, the reduced row echelon form $\mathbf{E}$ of $\mathbf{A}$ is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 13 & -4 & 11 \\ 0 & 1 & -4 & 3 & -4 \end{bmatrix}.$$

$$x_3 = r, \quad x_4 = s, \quad x_5 = t, \quad x_2 = 4r - 3s + 4t, \quad x_1 = -13r + 4s - 11t.$$

Therefore

$$\vec{u}_1 = (-13, 4, 1, 0, 0), \quad \vec{u}_2 = (4, -3, 0, 1, 0), \quad \vec{u}_3 = (-11, 4, 0, 0, 1).$$

26) Because

$$\vec{u}.\vec{v} = |\vec{u}||\vec{v}|\cos\theta,$$

it follows that $\vec{u}.\vec{v} = |\vec{u}||\vec{v}|$ if and only if $\cos\theta = 1$, in which case $\theta = 0$, so the two vectors are collinear.

28) If W is the orthogonal complement of V, then every vector in V is orthogonal to every vector in W. Hence V is contained in $W^\perp$. But it follows from $(\dim V + \dim V^\perp = n)$ that the two subspaces

[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

V and $W^{\perp}$ have the same dimension. Because one contains the other, they must therefore be the same subspace, so $W^{\perp} = V$ as desired.

30) If $\vec{u}.\vec{v} = 0$ and $\vec{u} + \vec{v} = \vec{0}$ then

$$0 = \vec{u}.(\vec{u} + \vec{v}) = \vec{u}.\vec{u} + \vec{u}.\vec{v} = \vec{u}.\vec{u}$$

so it follows that $\vec{u} = \vec{0}$ and then $\vec{u} + \vec{v} = \vec{0}$ implies that $\vec{v} = \vec{0}$ also.

31) We want to show that any linear combination of vectors $\vec{u}_1, \vec{u}_2, ..., \vec{u}_p$ of vectors in S is orthogonal to every linear combination of vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_q$ in T. But if each $\vec{u}_i$ is orthogonal to each $\vec{v}_j$ so $\vec{u}_i.\vec{v}_j = 0$, then it follows that

$$(a_1\vec{u}_1 + a_2\vec{u}_2 + ... + a_p\vec{u}_p)(b_1\vec{v}_1 + b_2\vec{v}_2 + ... + b_q\vec{v}_q) = \sum_{i=1}^{p}\sum_{j=1}^{q} a_i b_j \vec{u}_i \vec{v}_j = 0,$$

so we see that the two linear combinations are orthogonal, as desired.

32) Suppose that the linear combination

$$a_1\vec{u}_1 + a_2\vec{u}_2 + b_1\vec{v}_1 + b_2\vec{v}_2 = \vec{0},$$

and we want to deduce that all four coefficients a_1, a_2, b_1, b_2 must necessarily be zero. For this purpose, let

$$\vec{u} = a_1\vec{u}_1 + a_2\vec{u}_2, \quad \text{and} \quad \vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2.$$

Then the vectors $\vec{u}$ and $\vec{v}$ are orthogonal by Problem 31, so by Problem 30 the fact that $\vec{u} + \vec{v} = \vec{0}$ implies that

$$\vec{u} = a_1\vec{u}_1 + a_2\vec{u}_2 = \vec{0}, \quad \text{and} \quad \vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2 = \vec{0}.$$

Now the assumed linear independence of $\{\vec{u}_1, \vec{u}_2\}$ implies that $a_1 = a_2 = 0$, and the assumed linear independence of $\{\vec{v}_1, \vec{v}_2\}$ implies that $b_1 = b_2 = 0$, Thus we conclude that all four coefficients are zero, as desired.

33) This is the same as Problem 32 (which was solved in the class), except with

$$\vec{u} = a_1\vec{u}_1 + a_2\vec{u}_2 + ... + a_k\vec{u}_k, \qquad \text{and} \qquad \vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2 + ... + b_m\vec{v}_m.$$

34) It follows immediately from Problem 33 and from $(\dim V + \dim V^{\perp} = n)$ that the union of a basis for the subspace V and a basis for its orthogonal complement $V^{\perp}$ is a linearly independent set of n vectors, and is therefore a basis for the n-dimensional vector space $\mathbf{R}^n$.

35) This is one of the fundamental theorems of linear algebra. The nonhomogeneous system

$$\mathbf{A}\vec{x} = \vec{b}$$

is consistent if and only if the vector $\vec{b}$ is in the subspace $\text{Col}(\mathbf{A}) = \text{Row}(\mathbf{A}^T)$. But $\vec{b}$ is in $\text{Row}(\mathbf{A}^T)$ if and only if $\vec{b}$ is orthogonal to the orthogonal complement of $\text{Row}(\mathbf{A}^T)$. But $\text{Row}(\mathbf{A}^T)^{\perp} = \text{Null}(\mathbf{A}^T)$, which is the solution space of the homogeneous system

$$\mathbf{A}^T\vec{y} = \vec{0}$$

Thus we have proved (as desired) that the nonhomogeneous system $\mathbf{A}\vec{x} = \vec{b}$ has a solution if and only if the constant vector $\vec{b}$ is orthogonal to every solution $\vec{y}$ of the nonhomogeneous system $\mathbf{A}^T\vec{y} = \vec{0}$.