BİLKENT UNIVERSITY
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MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 14

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Homework problems from the 2$^{\text{nd}}$ Edition, SECTION 4.4

3(3))[2] Any four vectors in $\mathbf{R}^3$ are L.D., so the given vectors do not form a basis for $\mathbf{R}^3$.

8(8)) Since $\det\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4\end{bmatrix} = 66 \neq 0$, the given four vectors are L.I. and hence they form a basis for $\mathbf{R}^4$.

10(10)) The single equation $y - z = 0$ is already a system in reduced echelon form, with free variables x and z. With the choice of $x = s,\ y = z = t$, we get the solution vector

$$(x, y, z) = (s, t, t) = s(1, 0, 0) + t(0, 1, 1) = s\vec{v}_1 + t\vec{v}_2.$$

So the given plane is a 2-dimensional subspace of $\mathbf{R}^3$ with the basis consisting of the vectors $\vec{v}_1,\ \vec{v}_2$.

12(12)) The typical vector in $\mathbf{R}^4$ of the form (a, b, c, d) with $a = b + c + d$ can be written as

$$\vec{v} = (b + c + d,\ b,\ c,\ d) = b(1, 1, 0, 0) + c(1, 0, 1, 0) + d(1, 0, 0, 1).$$

Hence the subspace consisting of all such vectors in 3-dimensional with basis consisting of the vectors $\vec{v}_1 = (1, 1, 0, 0),\ \vec{v}_2 = (1, 0, 1, 0),\ \vec{v}_3 = (1, 0, 0, 1)$.

13(13)) The typical vector in $\mathbf{R}^4$ of the form (a, b, c, d) with $a = 3c$ and $b = 4d$ can be written as

$$\vec{v} = (3c,\ 4d,\ c,\ d) = c(3, 0, 1, 0) + d(0, 4, 0, 1).$$

Hence the subspace consisting of all such vectors in 2-dimensional with basis consisting of the vectors $\vec{v}_1 = (3, 0, 1, 0),\ \vec{v}_2 = (0, 4, 0, 1)$.

19(19)) The coefficient matrix $\mathbf{A}$ and its reduced row echelon for $\mathbf{E}$ are

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & -8 & -5 \\ 2 & 1 & -4 & 11 \\ 1 & 3 & 3 & 13 \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

respectively. Therefore $x_3 = s$ and $x_4 = t$ are free variables and $x_1 = 3s - 4t$ and $x_2 = -2s - 3t$. Hence the solution vector is

$$\vec{x} = s(3, -2, 1, 0) + t(-4, -3, 0, 1).$$

Thus the solution space of the given system is 2-dimensional with the basis consisting of the vectors $\vec{v}_1 = (3, -2, 1, 0),\ \vec{v}_2 = (-4, -3, 0, 1)$.

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

[2] The number in the parenthesis denotes the problem number in the International Edition of the textbook

23(23)) The coefficient matrix **A** and its reduced row echelon for **E** are

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 13 & 14 \\ 2 & 5 & 11 & 12 \\ 2 & 7 & 17 & 19 \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

respectively. Therefore $x_3 = s$ is the free variables and $x_1 = 2s$ and $x_2 = -3s,\ x_4 = 0$. Hence the solution vector is $\vec{x} = s(2, -3, 1, 0)$. Thus the solution space of the given system is 1-dimensional with the basis consisting of the vector $\vec{v}_1 = (2, -3, 1, 0)$.

26(26)) The coefficient matrix **A** and its reduced row echelon for **E** are

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & -3 & 11 & 10 \\ 5 & 8 & 2 & -2 & 7 \\ 2 & 5 & 0 & -1 & 14 \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 2 & -3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & -2 & -5 \end{bmatrix}$$

respectively. Therefore $x_4 = s$ and $x_5 = t$ are free variables and $x_1 = -2s + 3t,\ x_2 = s - 4t$ and $x_3 = 2s + 5t$. Hence the solution vector is

$$\vec{x} = s(-2, 1, 2, 1, 0) + t(3, -4, 5, 0, 1).$$

Thus the solution space of the given system is 2-dimensional with the basis consisting of the vectors $\vec{v}_1 = (-2, 1, 2, 1, 0),\ \vec{v}_2 = (3, -4, 5, 0, 1)$.

28(28)) If the n vectors in S were not L.I., then some one of them would be a linear combination of the others. These remaining $n - 1$ vectors would then span the n-dimensional vector space V, which is impossible. Therefore the spanning set S is also L.I., and therefore is a basis for V.

29(29)) Suppose $c\vec{v} + c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_k\vec{v}_k = \vec{0}$. Then $c = 0$ because , otherwise we could solve for $vecv$ as a linear combination of the vectors $\vec{v}_1,\ \vec{v}_2, ..., \vec{v}_k$. But this is impossible, because $\vec{v}$ is not in the subspace W spanned by $\vec{v}_1,\ \vec{v}_2, ..., \vec{v}_k$. It follows that $c_1\vec{v}_1 + c_2\vec{v}_2 + ... + c_k\vec{v}_k = \vec{0}$, which implies that $c_1 = c_2 = ... = c_k = 0$ also, because the vectors $\vec{v}_1,\ \vec{v}_2, ..., \vec{v}_k$ are L.I. Hence we have shown that the $k + 1$ vectors $\vec{v},\ \vec{v}_1,\ \vec{v}_2, ..., \vec{v}_k$ are L.I.

31(31)) If $\vec{v}_{k+1}$ is a L.C. of the vectors $\vec{v}_1, \vec{v}_2, ...\vec{v}_k$, then obviously every linear combination of vectors $\vec{v}_1, \vec{v}_2, ...\vec{v}_k, \vec{v}_{k+1}$, is also a L.C. of $\vec{v}_1, \vec{v}_2, ...\vec{v}_k$. But the former set of $k+1$ vectors spans V, so the later set of k vectors also spans V.

34(34)) If the minimal spanning set S for V were not L.I. then, by the problem # 28, some vector in S would be a linear combination of the others. Then the set is obtained from the minimal spanning set S by deleting this dependent vector would be a smaller spanning set for S (which is impossible). Hence the spanning set S is also L.I. set, and therefore is a basis for V.