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MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 13

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Homework problems from the 2nd Edition, SECTION 4.3

3(3))[2] Clearly any three vectors in $\mathbf{R}^2$ are L.D. Since, the L.C. $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ reduces to a homogenous system of two equations in three unknowns c_1, c_2 and c_3 with the coefficient matrix $\mathbf{A} = \begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix}$. Such system has a nontrivial solution.

6(6)) The L.C. $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ yields

$$c_1(1,0,0) + c_2(1,1,0) + c_3(1,1,1) = (c_1 + c_2 + c_3, c_2 + c_3, c_3) = (0,0,0).$$

Therefore $c_1 = c_2 = c_3 = 0$. Hence the given vectors are L.I.

11(11)) Set up the linear system to be solved for the linear combination coefficients $\{c_i\}$, $i = 1, 2$ and then show that the reduction of its augmented matrix $\mathbf{A}$ to reduced echelon form $\mathbf{E}$. In this problem, the L.C. is $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{w}$ and which is equivalent to the non homogenous system with the following augmented matrix

$$\mathbf{A} = \begin{bmatrix} 7 & 3 & 1 \\ -6 & -3 & 0 \\ 4 & 2 & 0 \\ 5 & 3 & -1 \end{bmatrix}$$

and its reduced row echelon form $\mathbf{E}$ is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the system of four equations in two unknowns has the unique solution $c_1 = 1,\ c_2 = -2$, so $\vec{w} = \vec{v}_1 - 2\vec{v}_2$.

16(16)) Similar to the previous problem, the L.C. is $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{w}$ and which is equivalent to the non homogenous system with the following augmented matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 1 & 7 \\ 0 & 1 & 3 & 7 \\ 3 & 3 & -1 & 9 \\ 1 & 2 & 3 & 11 \end{bmatrix}$$

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

[2] The number in the parenthesis denotes the problem number in the International Edition of the textbook

and its reduced row echelon form $\mathbf{E}$ is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, the system of four equations in three unknowns has the unique solution $c_1 = 6,\ c_2 = -2,\ c_3 = 3$, so $\vec{w} = 6\vec{v}_1 - 2\vec{v}_2 + 3\vec{v}_3$.

21(21)) The L.C. $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ is equivalent the homogenous system of equations in $\{c_i\},\ i = 1, 2, 3$ with the coefficient matrix $\mathbf{A} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$.

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

The reduced row echelon form $\mathbf{E}$ of $\mathbf{A}$ is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The system of 4 equations in 3 unknowns has a one dimensional solution space. If we let $c_3 = -1$ then $c_1 = 1$ and $c_2 = -2$. Therefore, $\vec{v}_1 - 2\vec{v}_2 - \vec{v}_3 = \vec{0}$.

23(23)) Since $\vec{v}_1$ and $\vec{v}_2$ are L.I., the vector equation

$$c_1\vec{u}_1 + c_2\vec{u}_2 = c_1(\vec{v}_1 + \vec{v}_2) + c_2(\vec{v}_1 - \vec{v}_2) = \vec{0}$$

yields the homogenous system

$$c_1 + c_2 = 0, \qquad c_1 - c_2 = 0.$$

Therefore, $c_1 = c_2 = 0$, and $\vec{u}_1$ and $\vec{u}_2$ are L.I.

26(26)) Since the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are L.I. the vector equation

$$c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 = c_1(\vec{v}_2 + \vec{v}_3) + c_2(\vec{v}_1 + \vec{v}_3) + c_3(\vec{v}_1 + \vec{v}_2) = \vec{0}$$

yields the homogenous system of three equations in c_1, c_2, c_3 with the coefficient matrix $\mathbf{A}$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The reduced row echelon form $\mathbf{E}$ of $\mathbf{a}$ is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So $c_1 = c_2 = c_3 = 0$, therefore the vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are L.I.

29(29)) If some subset of S were L.D. , the Problem # 28 would imply immediately that S itself is L.D. (contrary to hypothesis).

31(31)) If S is contained in span(T), then every vector in S is a L.C. of vectors in T. Hence every vector in span(S) is a L.C. of linear combinations of vectors in T. Therefore every vector in span(S) is a linear combination of vectors in T, and therefore is itself in span(T). Thus span(S) is a subset of span(T).

33(33)) The determinant of the $k \times k$ identity matrix is nonzero, so by the theorem (Theorem # 3 in the text book), $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$ are L.I.

34(34)) If the vectors $\vec{v}_1, \vec{v}_2, ...\vec{v}_k$ are L.I. then by the theorem (Theorem #2 in the text book) the matrix $\mathbf{A} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & ... & \vec{v}_k \end{bmatrix}$ is nonsingular. If $\mathbf{B}$ is another nonsingular $n \times n$ matrix, then the product $\mathbf{AB}$ is also nonsingular, and therefore (by Theorem #2 in the text book) has L.I. column vectors.

35(35)) Since the vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_k$ are L.I. by the theorem (Theorem # 3 in the text book) some $k \times k$ submatrix $\mathbf{A}_0$ of $\mathbf{A}$ has nonzero determinant. Let $\mathbf{A}_0$ consist of the rows $i_1, i_2, ..., i_k$ of the matrix $\mathbf{A}$, and let $\mathbf{C}_0$ denote the $k \times k$ submatrix consisting of the same rows of the product matrix $\mathbf{C} = \mathbf{AB}$. Then $\mathbf{C}_0 = \mathbf{A}_0\mathbf{B}$, so $(\det \mathbf{C}_0) = (\det \mathbf{A}_0)(\det \mathbf{B}) \neq 0$ because (by the hypothesis) the $k \times k$ matrix $\mathbf{B}$ is also nonsingular. Therefore by the theorem (Theorem # 3 in the text book) the column vectors of $\mathbf{AB}$ are L.I.