BİLKENT UNIVERSITY
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MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 12

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Homework problems from the 2nd Edition, SECTION 4.2

3(3))[2] A typical vector in W is of the form $\vec{x} = (x_1, 1, x_3)$. Since scalar multiple $c\vec{x} = (cx_1, c, cx_3)$ is not in W unless $c = 1$. Hence W is not closed under multiplication by scalar, and therefore W is not a subspace of $\mathbf{R}^3$.

7(7)) Note that the vectors $\vec{x} = (1, 1)$ and $\vec{y} = (1, -1)$ are in W. But their sum $\vec{x} + \vec{y} = (2, 0)$ is not in W, because $|2| \neq |0|$. Hence, W is not a subspace of $\mathbf{R}^2$.

8(8)) Since $x_1^2 + x_2^2 = 0$, W is simply the zero subspace $\{\vec{0}\ \}$ of $\mathbf{R}^2$.

10(10)) The vectors $\vec{x} = (1, 0)$ and $\vec{y} = (0, 1)$ are in W. But their sum $\vec{x} + \vec{y} = (1, 1)$ is not in W, because $|1| + |1| = 2 \neq 1$. Hence, W is not a subspace of $\mathbf{R}^2$.

11(11)) Let $\vec{x} = (x_1, x_2, x_3, x_4)$ and $\vec{y} = (y_1, y_2, y_3, y_4)$ be the vectors in W, so

$$x_1 + x_2 = x_3 + x_4, \qquad \text{and} \qquad y_1 + y_2 = y_3 + y_4.$$

Then their sum

$$\vec{s} = \vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) = (s_1, s_2, s_3, s_4)$$

satisfy the condition $s_1 + s_2 = s_3 + s_4$ and thus $\vec{s} \in W$. Similarly, let $\vec{m} = c\vec{x} = (cx_1, cx_2, cx_3, cx_4) = (m_1, m_2, m_3, m_4)$ satisfies the condition

$$m_1 + m_2 = cx_1 + cx_2 = c(x_1 + x_2) = c(x_3 + x_4) = cx_3 + cx_4 = m_3 + m_4.$$

and hence $\vec{m} \in W$. Therefore, W is a subspace of $\mathbf{R}^4$.

17(17)) The reduced row echelon form $\mathbf{E}$ of the coefficient matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 8 & -1 \\ 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \end{bmatrix}$$

is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

[2] The number in the parenthesis denotes the problem number in the International Edition of the textbook

Thus $x_3 = s$ and $x_4 = t$ are free variables. By the back substitution we find that $x_1 = s - 2t$ and $x_2 = -3s + t$. Therefore, the solution vector is $\vec{x} = (x_1, x_2, x_3, x_4) = (s - 2t, -3s + t, s, t) = s(1, -3, 1, 0) + t(-2, 1, 0, 1) = s\vec{u} + t\vec{v}$.

21(**21**)) Similar to the previous problem, the reduced row echelon form $\mathbf{E}$ of the coefficient matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 7 & 2 & -3 \\ 2 & 7 & 1 & -4 \\ 3 & 5 & -1 & -5 \end{bmatrix}$$

is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

Thus $x_4 = t$ is free variable. By the back substitution we find that $x_1 = -3t$, $x_2 = 2t$ and $x_4 = -4t$. Therefore, the solution vector is $\vec{x} = (x_1, x_2, x_3, x_4) = (-3t, 2t, -4t, t) = t(-3, 2, -4, 1) = t\vec{u}$.

25(**25**)) If W is a subspace, then it contains the scalar multiples $a\vec{u}$ and $b\vec{v}$, and hence contains their sum $a\vec{u} + b\vec{v}$. Conversely, if the subset W is closed under taking linear combinations of pairs of vectors, then it contains $(1)\vec{u} + (1)\vec{v} = \vec{u} + \vec{v}$ and $(c)\vec{u} + (0)\vec{v} = c\vec{u}$, and hence is a subspace.

27(**27**)) Let $a_1\vec{u} + b_1\vec{v}$ and $a_2\vec{u} + b_2\vec{v}$ be two vectors in $W = \{a\vec{u} + b\vec{v}\}$. Then the sum

$$(a_1\vec{u} + b_1\vec{v}) + (a_2\vec{u} + b_2\vec{v}) = (a_1 + a_2)\vec{u} + (b_1 + b_2)\vec{v}$$

and the scalar multiple $c(a_1\vec{u} + b_1\vec{v}) = (ca_1)\vec{u} + (cb_1)\vec{v}$ are given scalar multiples of $\vec{u}$ and $\vec{v}$. Hence W is a subspace.

28(**28**)) If $\vec{u} \in W$ and $\vec{v} \in W$, then $A\vec{u} = k\vec{u}$. It follows that

$$\mathbf{A}(a\vec{u} + b\vec{v}) = a(\mathbf{A}\vec{u}) + b(\mathbf{A}\vec{v}) = a(k\vec{u}) + b(k\vec{v}) = k(a\vec{u} + b\vec{v}),$$

so the linear combination $a\vec{u} + b\vec{v}$ of $\vec{u}$ and $\vec{v}$ is also in W. Hence W is a subspace.

29(**29**)) If $\mathbf{A}\vec{x}_0 = \vec{b}$ and $y = x - x_0$ then

$$\mathbf{A}\vec{y} = \mathbf{A}(\vec{x} - \vec{x}_0) = \mathbf{A}\vec{x} - \vec{b} = \vec{0}.$$

Hence $\mathbf{A}\vec{y} = \vec{0}$ if and only if $\mathbf{A}\vec{x} = \vec{0}$.