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MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 11

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Homework problems from the 2nd Edition, SECTION 4.1

3(3))[2]

$$|\vec{a}-\vec{b}| = |(2\vec{i}-3\vec{j}+5\vec{k})-(5\vec{i}+3\vec{j}-7\vec{k})| = |-3\vec{i}-6\vec{j}+12\vec{k}| = \sqrt{189} = 3\sqrt{21}.$$

$$2\vec{a}+\vec{b} = 2(2\vec{i}-3\vec{j}+5\vec{k})+(5\vec{i}+3\vec{j}-7\vec{k}) = (4\vec{i}-6\vec{j}+10\vec{k})+(5\vec{i}+3\vec{j}-7\vec{k}) = 9\vec{i}-3\vec{j}+3\vec{k}.$$

Similarly,

$$3\vec{a}-4\vec{b} = -14\vec{i}-21\vec{j}+43\vec{k}.$$

5(5)) Since $\vec{v}=\frac{1}{2}\vec{u}$, the vectors $\vec{u}$ and $\vec{v}$ are L.D.

7(7)) $a\vec{u}+b\vec{v} = a(0,2)+b(3,0) = (3b,2a) = \vec{0}$ implies that $a=b=0$, so the vectors $\vec{u}$ and $\vec{v}$ are L.I.

11(11)) Let

$$\vec{w} = a\vec{u}+b\vec{v},$$

and the L.C. is equivalent to the following nonhomogenous linear system:

$$\begin{bmatrix}5 & 2\\ 7 & 3\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}1\\ 1\end{bmatrix}.$$

Therefore, $a=1,\quad b=-2$, so, $\vec{w}=\vec{u}-2\vec{v}$.
13(13)) Similar to the previous problem,

$$\begin{bmatrix}7 & 3\\ 5 & 4\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}5\\ -2\end{bmatrix}.$$

Therefore, $a=2,\quad b=-3$, so, $\vec{w}=2\vec{u}-3\vec{v}$.

15(15)) We should calculate the determinant of the matrix $A=\begin{bmatrix}\vec{u} & \vec{v} & \vec{w}\end{bmatrix}$, whether the three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ are L.D. (det $=0$) or L.I. (det $\neq 0$).

$$\det\begin{bmatrix}3 & 5 & 8\\ -1 & 4 & 3\\ 2 & -6 & -4\end{bmatrix} = 0.$$

[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

[2] The number in the parenthesis denotes the problem number in the International Edition of the textbook

So the given vectors are L.D.

18(18)) Similar to the previous problem

$$\det \begin{bmatrix} 1 & 4 & 3 \\ 1 & 3 & -2 \\ 0 & 1 & -4 \end{bmatrix} = 9 \neq 0.$$

So the given vectors are L.I.

20(20)) In this problem, we should solve the homogenous system $\mathbf{A}\vec{x} = \vec{0}$ by reducing the coefficient matrix $\mathbf{A} = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$ to echelon form $\mathbf{E}$. If the system has only trivial solution $a = b = c = 0$, then the vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ are L.I. Otherwise, a nontrivial solution implies that the vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ are L.D. In this problem

$$\mathbf{A} = \begin{bmatrix} 5 & 2 & 4 \\ 5 & 3 & 1 \\ 4 & 1 & 5 \end{bmatrix}$$

and its reduced row echelon form is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the system has nontrivial solution $a = -2,\ b = 3,\ c = 1$ and $-2\vec{u} + 3\vec{v} + \vec{w} = \vec{0}$, so the vectors are L.D.

22(22)) Similar to the previous problem, in this problem

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

and its reduced row echelon form is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, the system has only trivial solution $a = b = c = 0$, so the vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ are L.I.

23(23)) Similar to the previous problem, in this problem

$$\mathbf{A} = \begin{bmatrix} 2 & 5 & 2 \\ 0 & 4 & -1 \\ 3 & -2 & 1 \end{bmatrix}$$

and its reduced row echelon form is

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, the system has only trivial solution $a = b = c = 0$, so the vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ are L.I.

25(25)) In this problem we should solve the non-homogenous linear system $\mathbf{A}\vec{x} = \vec{t}$ with the coefficient matrix $\mathbf{A} = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$ and the augmented coefficient matrix $[\mathbf{A}|\,\vec{t}\,] = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} & \vec{t} \end{bmatrix}$. If the reduced row echelon form of the augmented coefficient matrix is $\mathbf{E}$, the solution vector $\vec{x} = [a\ b\ c]^T$ appears as the final column of $\mathbf{E}$, and provides us with the desired linear combination $\vec{t} = a\vec{u} + b\vec{v} + c\vec{w}$. In this problem:

$$[\mathbf{A}|\,\vec{t}\,] = \begin{bmatrix} 1 & 3 & 1 & 2 \\ -2 & 0 & -1 & -7 \\ 2 & 1 & 2 & 9 \end{bmatrix}, \quad \text{and} \quad \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Thus, $a = 2,\ b = -1,\ c = 3$, so $\vec{t} = 2\vec{u} - \vec{v} + 3\vec{w}$.

26(**26**)) Similar to the previous problem, in this case

$$[\mathbf{A}|\,\vec{t}\,] = \begin{bmatrix} 5 & 1 & 5 & 5 \\ 2 & 5 & -3 & 30 \\ -2 & -3 & 4 & -21 \end{bmatrix}, \quad \text{and} \quad \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Thus, $a = 1,\ b = 5,\ c = -1$, so $\vec{t} = \vec{u} + 5\vec{v} - \vec{w}$.

32(**32**)) If (x, y, z) and (u, v, w) are in V, then

$$z + w = (2x + 3y) + (2u + 3v) = 2(x + u) + 3(y + v),$$

so their sum $(x + u, y + v, z + w)$ is in V. Similarly,

$$cz = c(2x + 3y) = 2(cx) + 3(cy),$$

so the scalar multiple (cx, cy, cz) is in V. So V is a subspace of $\mathbf{R}^3$.

35(**35**)) V is closed under the addition of vectors. However, $(0, 0, 1) \in V$, but $(-1)(0, 0, 1) = (0, 0, -1) \notin V$, so V is not closed under the multiplication by scalar. So V is not a subspace of $\mathbf{R}^3$.

36(**36**)) $(1, 1, 1) \in V$, but

$$2(1, 1, 1) = (1, 1, 1) + (1, 1, 1) = (2, 2, 2)$$

is not, so V is not closed either addition of vectors and under the multiplication by scalar. So V is not a subspace of $\mathbf{R}^3$.

38(**38**)) Suppose $\vec{u}$ and $\vec{v}$ are vectors in the subspace V of $\mathbf{R}^3$ and a and b are the scalars. Then $a\vec{u}$ and $b\vec{v}$ are in V because V is closed under multiplication by scalars. But then it follows that the linear combination $a\vec{u} + b\vec{v}$ is in V because V is closed under addition of vectors.

41(**41**)) If $\vec{u}$ and $\vec{v}$ are in the intersection V of the subspaces V_1 and V_2, then their sum $\vec{u} + \vec{v}$ is in V_1 because both vectors are in V_1, and $\vec{u} + \vec{v}$ is in V_2 because both vectors are in V_2. Therefore, $\vec{u} + \vec{v}$ is in V and thus V is closed under addition of vectors. Similarly, the intersection V is closed under multiplication by scalars, and is therefore itself a subspace.