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MATH 225, LINEAR ALGEBRA and DIFFERENTIAL EQUATIONS,
Solution of Homework set[1] # 10

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Homework problems from the 2nd Edition, SECTION 3.6

23(23))[2] The coefficient matrix **A** is

$$\mathbf{A} = \begin{bmatrix} 17 & 7 \\ 12 & 5 \end{bmatrix}$$

and $\det \mathbf{A} = 1$. Therefore by the Cramer's rule

$$x = \frac{1}{\det \mathbf{A}} \begin{vmatrix} 6 & 7 \\ 4 & 5 \end{vmatrix} = 2.$$

and

$$x = \frac{1}{\det \mathbf{A}} \begin{vmatrix} 17 & 6 \\ 12 & 4 \end{vmatrix} = -4.$$

28(28)) The coefficient matrix **A** is

$$\mathbf{A} = \begin{bmatrix} 5 & 4 & -2 \\ 2 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}$$

and $\det \mathbf{A} = 35$. By the Cramer's rule

$$x_1 = \frac{1}{\det \mathbf{A}} \begin{vmatrix} 4 & 4 & -2 \\ 2 & 0 & 3 \\ 1 & -1 & 1 \end{vmatrix} = 4/7, \quad x_2 = \frac{1}{\det \mathbf{A}} \begin{vmatrix} 5 & 4 & -2 \\ 2 & 2 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 3/7, \quad x_3 = \frac{1}{\det \mathbf{A}} \begin{vmatrix} 5 & 4 & 4 \\ 2 & 0 & 2 \\ 2 & -1 & 1 \end{vmatrix} = 2/7.$$

31(31)) The coefficient matrix **A** is

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -5 \\ 4 & -5 & 3 \\ -2 & 1 & 1 \end{bmatrix}$$

and $\det \mathbf{A} = 14$. By the Cramer's rule

$$x_1 = \frac{1}{\det \mathbf{A}} \begin{vmatrix} -3 & 0 & -5 \\ 3 & -5 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -\frac{8}{7}, \; x_2 = \frac{1}{\det \mathbf{A}} \begin{vmatrix} 2 & -3 & -5 \\ 4 & 3 & 3 \\ -2 & 1 & 1 \end{vmatrix} = -\frac{10}{7}, \; x_3 = \frac{1}{\det \mathbf{A}} \begin{vmatrix} 2 & 0 & -3 \\ 4 & -5 & 3 \\ -2 & 1 & 1 \end{vmatrix} = \frac{1}{7}.$$

35(35)) First calculate the minors $M_{ij},\ i,j = 1,2,3.$

$$M_{11} = -15, \quad M_{12} = -10, \quad M_{13} = 15, \quad M_{21} = -25, \quad M_{22} = -5, \quad M_{23} = 25,$$

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[1] I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. **You are responsible to check all the solutions and correct the errors if there is any.** If you find any errors and/or misprints, please notify me.

[2] The number in the parenthesis denotes the problem number in the International Edition of the textbook

$$M_{31} = -26, \quad M_{32} = -8, \quad M_{33} = 19.$$

Therefore the cofactor, since $A_{ij} = (-1)^{i+j} M_{ij}$

$$A_{11} = -15, \quad A_{12} = 10, \quad A_{13} = 15, \quad A_{21} = 25, \quad A_{22} = -5, \quad A_{23} = -25,$$

$$A_{31} = -26, \quad A_{32} = 8, \quad A_{33} = 19.$$

Hence

$$\left[A_{ij}\right] = \begin{bmatrix} -15 & 10 & 15 \\ 25 & -5 & -25 \\ -26 & 8 & 19 \end{bmatrix}$$

and the transpose of $\left[A_{ij}\right]$

$$\left[A_{ij}\right]^T = \begin{bmatrix} -15 & 25 & -26 \\ 10 & -5 & 8 \\ 15 & -25 & 19 \end{bmatrix}$$

and $\det \mathbf{A} = 35$,

$$\mathbf{A^{-1}} = \frac{1}{35} \begin{bmatrix} -15 & 25 & -26 \\ 10 & -5 & 8 \\ 15 & -25 & 19 \end{bmatrix}.$$

37(37)) Similar to previous problem first calculate M_{ij}, $i, j = 1, 2, 3.$ and then $\left[A_{ij}\right]$, and take the transpose of $\left[A_{ij}\right]$. The theorem implies that $\mathbf{A^{-1}} = \frac{\left[A_{ij}\right]^T}{\det \mathbf{A}}$. Therefore

$$\det \mathbf{A} = 29, \qquad \text{and,} \quad \mathbf{A^{-1}} = \frac{1}{29} \begin{bmatrix} 11 & -14 & -15 \\ -17 & 19 & 10 \\ 18 & -15 & -14 \end{bmatrix}.$$

39(39)) Similar to previous problem first calculate M_{ij}, $i, j = 1, 2, 3.$ and then $\left[A_{ij}\right]$, and take the transpose of $\left[A_{ij}\right]$. The theorem implies that $\mathbf{A^{-1}} = \frac{\left[A_{ij}\right]^T}{\det \mathbf{A}}$. Therefore

$$\det \mathbf{A} = 37, \qquad \text{and,} \quad \mathbf{A^{-1}} = \frac{1}{37} \begin{bmatrix} -21 & -1 & -13 \\ 4 & 9 & 6 \\ -6 & 5 & -9 \end{bmatrix}.$$

41(41)) If

$$\mathbf{A} = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \end{bmatrix}, \qquad \text{and,} \quad \mathbf{B} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}$$

in terms of two row vectors of $\mathbf{A}$ and two column vectors of $\mathbf{B}$, then

$$\mathbf{AB} = \begin{bmatrix} \vec{a}_1\vec{b}_1 & \vec{a}_1\vec{b}_2 \\ \vec{a}_2\vec{b}_1 & \vec{a}_2\vec{b}_2 \end{bmatrix}$$

so

$$(\mathbf{AB})^T = \begin{bmatrix} \vec{a}_1\vec{b}_1 & \vec{a}_2\vec{b}_1 \\ \vec{a}_1\vec{b}_2 & \vec{a}_2\vec{b}_2 \end{bmatrix} = \begin{bmatrix} \vec{b}_1{}^T \\ \vec{b}_2{}^T \end{bmatrix} \begin{bmatrix} \vec{a}_1{}^T & \vec{a}_2{}^T \end{bmatrix} = \mathbf{B}^T \mathbf{A}^T.$$

55(55)) If either $\mathbf{AB} = \mathbf{I}$ or $\mathbf{BA} = \mathbf{I}$ is given, then it follows from the problem # 54 that $\mathbf{A}$ and $\mathbf{B}$ are both invertible because their product is invertible. Hence, $\mathbf{A}^{-1}$ exists. So if (for instance) it is

$\mathbf{AB} = \mathbf{I}$ that is given, then multiplication by $\mathbf{A}^{-1}$ on the right yields $\mathbf{B} = \mathbf{A}^{-1}$.

57(57)) Let

$$\mathbf{A} = \begin{bmatrix} a & d & f \\ 0 & b & e \\ 0 & 0 & c \end{bmatrix}$$

then

$$\mathbf{A}^{-1} = \frac{1}{abc} \begin{bmatrix} bc & -cd & de - bf \\ 0 & ac & -ae \\ 0 & 0 & ab \end{bmatrix}.$$

58(58)) The determinant of the coefficient matrix of the system for the unknowns $(\cos A, \cos B, \cos C)$ is

$$\det \mathbf{A} = \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} = 2abc.$$

Hence, by the Cramer's rule

$$\cos A = \frac{1}{2abc} \begin{vmatrix} a & c & b \\ b & 0 & a \\ c & a & 0 \end{vmatrix} = \frac{ab^2 - a^3 + ac^2}{2abc} = \frac{b^2 - a^2 + c^2}{2bc}.$$

So,

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

60(60)) In the case of 4×4, expansion w.r.t. the first row gives

$$\begin{vmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix},$$

So $B_4 = 2B_3 - B_2 = 2(4) - (3) = 5$. The general recursion relation $B_n = 2B_{n-1} - B_{n-2}$ results in the same way upon expansion along the first row.
If we assume inductively that

$$B_{n-1} = (n-1) + 1 = n \qquad \text{and} \qquad B_{n-2} = (n-2) + 1 = n - 1,$$

then the recursion relation of the previous part yields

$$B_n = 2B_{n-1} - B_{n-2} = 2(n) - (n-1) = n + 1.$$

61(61)) Subtraction of the first row from both the second and the third row gives

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2 - a^2 \\ 0 & c-a & c^2 - a^2 \end{vmatrix} = (b-a)(c^2 - a^2) - (c-a)(b^2 - a^2)$$

$$= (b-a)(c-a)(c+a) - (c-a)(b-a)(b+a)$$
$$= (b-a)(c-a)[(c-a) - (b+a)] = (b-a)(c-a)(c-b).$$

62(62)) Expansion of the 4×4 determinant defining $P(y)$ along its 4^{th} row yields

$$P(y) = y^3 \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} + ... = y^3 V(x_1, x_2, x_3) + \text{ lower degree terms in } \ y.$$

From the determinant definition of $P(y)$

$$P(x_1) = P(x_2) = P(x_3) = 0$$

the three roots of the cubic polynomial $P(y)$ are x_1, x_2, x_3. By factoring, we can write

$$P(y) = k(y - x_1)(y - x_2)(y - x_3)$$

for some constant k, and the calculation above implies that

$$k = V(x_1, x_2, x_3) = (x_3 - x_1)(x_3 - x_2)(x_2 - x_1).$$

Finally we can se that

$$V(x_1, x_2, x_3, x_4) = P(x_4) = V(x_1, x_2, x_3).(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)$$
$$= (x_4 - x_1)(x_4 - x_2)(x_4 - x_3)(x_3 - x_1)(x_3 - x_2)(x_2 - x_1)$$

which is the desired formula for $V(x_1, x_2, x_3, x_4)$.