BİLKENT UNIVERSITY
Department of Mathematics

Date: 27 October 2007 NAME:.........**SOLUTION KEY**.................

Time: 10:00-12:00 STUDENT NO:.........**SOLUTION KEY**.................

Fall 2007-08 SECTION: 01 02 03 04 06 07

Math 225.01-07, Linear Algebra & Differential Eq. Midterm Exam # 1

1	2	3	4	5	TOTAL
15	20	15	20	30	100
15	20	15	20	30	100

(Do not write anything on the above table)

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1) ind the solution of the following initial value problem in *explicit form*, and determine the interval in which the solution of the initial value problem is defined:

$$\frac{dy}{dx} = \frac{2x}{y + x^2 y}, \qquad y(0) = -2.$$

(15 points)

The given D.E. is separable type

$$ydy = \frac{2x}{1+x^2}dx$$

Then

$$y^2 = 2\ln(1+x^2) + c, \qquad c = \text{integration constant}.$$

I.C. gives $c = 4$. So

$$y = \pm[2\ln(1+x^2) + 4]^{1/2}$$

and the I.C. indicates that we must pick the negative square root. Thus the solution is

$$y = -[2\ln(1+x^2) + 4]^{1/2}.$$

Since $1 + x^2 > 0$ for all x, the solutionis valid for $-\infty < x < \infty$.

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2) Show that if

$$M(x,y)dx + N(x,y)dy = 0$$

is homogenous type differential equation, then it has

$$\mu(x,y) = \frac{1}{xM(x,y) + yN(x,y)}$$

for an integrating factor in order to have an exact type differential equation.
That is, $\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$ is of exact type.

(20 points)

Lets verify that the given $\mu(x,y)$ is an integrating factor, i.e. verify that

$$\frac{M}{xM + yN}dx + \frac{N}{xM + yN}dy = 0,$$

is exact. Since the equation is homogenous type then

$$\frac{M(x,y)}{N(x,y)} = F(y/x).$$

So, multiply the numerator and denominator of the D.E. by $1/N$ to get

$$\frac{F}{xF + y}dx + \frac{1}{xF + y}dy = 0,$$

and then check for the exactness. We have

$$\frac{\partial}{\partial x}\left[\frac{1}{xF + y}\right] = -\frac{F + x(-y/x^2)F'}{(xF + y)^2} = -\frac{1}{(xF + y)^2}\left[F - \frac{y}{x}F'\right],$$

$$\frac{\partial}{\partial y}\left[\frac{F}{xF + y}\right] = \frac{1}{(xF + y)^2}\left[(xF + y)\frac{1}{x}F' - F(F' + 1)\right] = -\frac{1}{(xF + y)^2}\left[F - \frac{y}{x}F'\right].$$

where F' denotes the derivative of F w.r.t. y/x.

Hence the equation is exact and $\mu(x,y)$ is an integrating factor.

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3) Suppose that the augmented coefficient matrix for a system of linear equations reduces to

$$[\mathbf{A}|\,\mathbf{b}\,] = \begin{bmatrix} 1 & 2 & 3 & : & 4 \\ 0 & 1 & 2 & : & 3 \\ 0 & 0 & k^2-k-2 & : & k+1 \end{bmatrix},$$

for what values of k system has **(a)** no solution; **(b)** a unique solution; **(c)** infinitely solutions. If the solution exists, find it.

(5+5+5 points)

Note that $k^2 - k - 2 = (k-2)(k+1)$. Then

(a) No solution, when $k^2 - k - 2 = 0$ and $k + 1 \neq 0$. This happens when $k = 2$.

(b) A unique solution when $k^2 - k - 2 \neq 0$. That is, $k \neq -1$. Then the corresponding system is

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_2 + 2x_3 &= 3 \\ (k-2)x_3 &= 1. \end{aligned}$$

By back substitution, one obtains the following solution;

$$x_1 = -\frac{2k-5}{k-2}, \qquad x_2 = \frac{3k-8}{k-2}, \qquad x_3 = \frac{1}{k-2}, \qquad k \neq 2.$$

Therefore, the system has a unique when $k \neq -1$ and $k \neq 2$.

(c) Infinitely many solutions when $k^2 - k - 2 = 0$ and $k + 1 = 0$. This happens when $k = -1$. If we let $x_3 = t$ then the solution

$$x_1 = t - 2, \qquad x_2 = 3 - 2t, \qquad x_3 = t.$$

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4) By using *Gauss-Jordan* elimination find the solution set of the following linear system of equations:

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 0 \\ 2x_1 - x_2 + x_3 - x_4 &= 3 \\ x_1 + x_2 + x_3 + x_4 &= 2 \ . \end{aligned}$$

(20 points)

First form the augmented coefficient matrix:

$$[\mathbf{A}|\,\vec{b}\,] = \begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ 2 & -1 & 1 & -1 & 3 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}.$$

Then perform Gaussian elimination as follows:

$$\xrightarrow{-\mathbf{2R_1+R_2}} \begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ 0 & -3 & 3 & -3 & 3 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{-\mathbf{R_1+R_3}} \begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ 0 & -3 & 3 & -3 & 3 \\ 0 & 0 & 2 & 0 & 2 \end{bmatrix}$$

At this stage Gaussian elimination is completed and we have an echelon form for the augmented coefficient matrix. Now we should multiply the leading variable rows so that the leading variables are all 1:

$$\xrightarrow{-\mathbf{R_2/3}} \begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 2 & 0 & 2 \end{bmatrix} \xrightarrow{\mathbf{R_3/2}} \begin{bmatrix} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Now, we can start Jordan's upward elimination process:

$$\xrightarrow{-\mathbf{R_2+R_1}} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{-\mathbf{R_2+R_2}} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Now, we completed the Gauss-Jordan elimination and reached to the desired echelon form. As can be seen, x_1, x_2 and x_3 are leading variables and x_4 is the free variable. Hence the solution set can be parametrically written as: $x_4 = t$, $x_3 = 1$, $x_2 = -t$, $x_1 = 1$. The solution set can be written in the vector form as:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -t \\ 1 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} .$$

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5) Classify the the following differential equations.
e.g. Linear in y, Homogenous, Separable, Exact, ...

DO NOT find the solutions and write ONLY ONE answer in the box.

a) $\dfrac{dy}{dx} = -\dfrac{2xy + y^2 + 1}{x^2 + 2xy}$ — **Ans: Exact**

b) $xy' + xy = 1 - y$ — **Ans: Linear in** y

c) $xy' + y - y^2 e^{2x} = 0,$ — **Ans: Bernoulli**

d) $(e^x + 1)y' = y - ye^x$ — **Ans: Separable**

e) $xdy - ydx = (xy)^{1/2}dx$ — **Ans: Homogenous**

f) $\dfrac{dy}{dx} = \dfrac{y^3}{1 - 2xy^2},$ — **Ans: Linear in** x

($6 \times 5 = 30$ points)

You can use the space below for your own calculations