Generalized scaling reductions and Painlevé hierarchies

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Abstract

We give an alternative derivation of two Painlevé hierarchies. This is done by constructing generalized scaling reductions of the Korteweg-de Vries and dispersive water wave hierarchies. We also construct a generalized scaling reduction of Burgers hierarchy.

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1 Introduction

The derivation of Painlevé hierarchies is an area of research that has recently proved to be of great interest. One natural question that arises is that of how to undertake similarity reductions of hierarchies of completely integrable partial differential equations (PDEs) in such a way as to include lower-weight terms in the resulting ordinary differential equations (ODEs). In a recent paper [1] we considered accelerating-wave type reductions of integrable hierarchies. In the present paper we turn our attention to generalized scaling reductions.

It is well-known that the Korteweg-de Vries (KdV) equation,

$$U_{t_3} = U_{xxx} + 6UU_x, \tag{1.1}$$

admits the generalized scaling reduction

$$U = \frac{f(z)}{[6g_0t_3]^{2/3}} + d, \qquad z = \frac{x}{[6g_0t_3]^{1/3}} + \frac{d}{g_0}[6g_0t_3]^{2/3}, \tag{1.2}$$

where $g_0 \neq 0$ and d are arbitrary constants, to the ordinary differential equation (ODE)

$$f_{zzz} + 6ff_z + g_0(4f + 2zf_z) = 0, \tag{1.3}$$

and that this last integrates to give the so-called thirty-fourth Painlevé equation (equation XXXIV in [2]). It is also well-known that the KdV equation is the first non-trivial member of a hierarchy of ompletely integrable PDEs,

$$U_{t_{2n+1}} = \mathcal{R}^n[U]U_x, \qquad \mathcal{R}[U] = \partial_x^2 + 4U + 2U_x\partial_x^{-1}, \tag{1.4}$$

where we have labelled the flow times in the usual way. However, as far as we know, it is not yet been shown how a hierarchy of ordinary differential equations (ODEs) based on the thirty-fourth Painlevé equation (a thirty-fourth Painlevé hierarchy) can be derived from a corresponding KdV hierarchy using an appropriate extension of the above generalized scaling reduction. Here we show how this can be done. We also show how, similarly, a fourth Painlevé hierarchy can be obtained from a generalized scaling reduction of the dispersive water wave (DWW) hierarchy. Finally, we consider a generalized scaling reduction of Burgers hierarchy.

2 The Korteweg-de Vries case

First of all, let us recall that the recursion operator $\mathcal{R}[U]$ of the KdV hierarchy (1.4) is the quotient $\mathcal{R}[U] = \mathcal{B}_1[U]\mathcal{B}_0^{-1}[U]$ of the two Hamiltonian operators

$$\mathcal{B}_1[U] = \partial_x^3 + 4U\partial_x + 2U_x, \qquad \mathcal{B}_0[U] = \partial_x, \tag{2.1}$$

the KdV hierarchy being bi-Hamiltonian, that is, can be written in Hamiltonian form in two different ways:

$$U_{t_{2n+1}} = \mathcal{R}^n[U]U_x = \mathcal{B}_0[U]M_{n+1}[U] = \mathcal{B}_1[U]M_n[U], \tag{2.2}$$

where the quantities $M_n[U]$, defined by $M_0[U] = 1/2$ and by the recursion relation given by the last equality in equation (2.2),

$$M_0[U] = \frac{1}{2}, \qquad M_1[U] = U, \qquad M_2[U] = U_{xx} + 3U^2, \qquad \ldots \tag{2.3}$$

are the variational derivatives of a corresponding sequence of Hamiltonian densities. We will use the Hamiltonian operator $\mathcal{B}_1[U]$ and the quantities $M_n[U]$ later in this section, but not the Hamiltonian densities themselves.

We now turn to our result that a thirty-fourth Painlevé hierarchy can be derived from a generalized scaling reduction of an extension of the above KdV hierarchy, in which we include lower order flows with coefficients functions of t_{2n+1} to be determined. We present our results in the form of a Proposition. Let us begin by recalling the following Lemma [1].

Lemma 1. The change of variables $\tilde{U} = U + C$, where C is an arbitrary constant, in $\mathcal{R}^n[\tilde{U}]\tilde{U}_x$, yields

$$\mathcal{R}^n[\tilde{U}]\tilde{U}_x = \sum_{j=0}^{n} \alpha_{n,j} C^{n-j} \mathcal{R}^j[U]U_x, \tag{2.4}$$

where the coefficients $\alpha_{n,j}$ are determined recursively by

$$\begin{aligned} \alpha_{n,n} &= 1, && (2.5)\\ \alpha_{n,j} &= 4\alpha_{n-1,j} + \alpha_{n-1,j-1}, \qquad j = 1, \ldots, n-1, && (2.6)\\ \alpha_{n,0} &= \frac{4n+2}{n}\alpha_{n-1,0} && (2.7) \end{aligned}$$

and where $\alpha_{0,0} = 1$.

Proposition 1. There exists a choice of coefficient funtions $\beta_i(t_{2n+1})$ and of the function $c(t_{2n+1})$ such that the substitution

$$U = \frac{f(z)}{[2(2n+1)g_{n-1}t_{2n+1}]^{2/(2n+1)}} + d, \qquad z = \frac{x}{[2(2n+1)g_{n-1}t_{2n+1}]^{1/(2n+1)}} + c(t_{2n+1}), \tag{2.8}$$

where $g_{n-1} \neq 0$ and d are arbitrary constants, into the hierarchy

$$U_{t_{2n+1}} = \mathcal{R}^n[U]U_x + \sum_{i=1}^{n-1} \beta_i(t_{2n+1})\mathcal{R}^i[U]U_x \tag{2.9}$$

yields the generalized thirty-fourth Painlevé hierarchy

$$K[f](K[f])_{zz} - \frac{1}{2}((K[f])_z)^2 + 2f(K[f])^2 + \frac{1}{2}(g_{n-1} + \alpha_n)^2 = 0 \tag{2.10}$$

where α_n is an arbitrary constant and

$$K[f] = M_n[f] + \sum_{i=0}^{n-1} B_i M_i[f] + g_{n-1}z, \tag{2.11}$$

the coefficients B_i being arbitrary constants and where by $M_j[f]$ we denote the variational derivatives of the Hamiltonian densities of the KdV hierarchy with dependent variable f and independent variable z.

Proof. Using Lemma 1 we see that substituting (2.8) in (2.9) gives

$$\begin{aligned} \sum_{k=0}^{n} \gamma_k \mathcal{R}^k[f]f_z + \frac{g_{n-1}}{T^{2n+3}}(4f + 2zf_z) \equiv \sum_{j=0}^{n} \frac{\alpha_{n,j}d^{n-j}}{T^{2j+3}}\mathcal{R}^j[f]f_z + \sum_{i=1}^{n-1}\left(\beta_i \sum_{j=0}^{i} \frac{\alpha_{i,j}d^{i-j}}{T^{2j+3}}\mathcal{R}^j[f]f_z\right) \\ -\frac{1}{T^2}f_z c_{t_{2n+1}} - 2\frac{g_{n-1}}{T^{2n+3}}f_z c + \frac{g_{n-1}}{T^{2n+3}}(4f + 2zf_z) = 0 \end{aligned} \tag{2.12}$$

where $T = [2(2n+1)g_{n-1}t_{2n+1}]^{1/(2n+1)}$ and $\mathcal{R}[f] = \partial_z^2 + 4f + 2f_z\partial_z^{-1}$. We recall that each $\alpha_{i,i} = 1$, and so in particular $\gamma_n = 1/T^{2n+3}$.

We solve the equations

$$\gamma_k = B_k/T^{2n+3}, \qquad k = n-1, \dots, 1 \tag{2.13}$$

recursively for the coefficients β_k and the equation

$$\gamma_0 = B_0/T^{2n+3} \tag{2.14}$$

for c, where all B_k are constants. The resulting equation can be written

$$\mathcal{B}_1[f]K[f] = 0 \tag{2.15}$$

where $\mathcal{B}_1[f] = \partial_z^3 + 4f\partial_z + 2f_z$ and $K[f]$ is as given in (2.11). This last equation admits (2.10) as a first integral, where α_n is an arbitrary constant of integration.

Remark 1. Without loss of generality, we may set, using a shift on z, $B_0 = 0$ in (2.10), (2.11). This then gives the generalized thirty-fourth Painlevé hierarchy as defined in [3]. The case where all $B_k = 0$ gives the thirty-fourth Painlevé hierarchy as originally defined in [4, 5], obtained from the non-generalized scaling reduction ($d = 0$ and $c = 0$)of the standard KdV hierarchy (1.4).

Example 1. The fifth order KdV equation

$$U_{t_5} = \left(U_{xxxx} + 10UU_{xx} + 5U_x^2 + 10U^3\right)_x + \beta_1(t_5)\left(U_{xx} + 3U^2\right)_x \tag{2.16}$$

admits the generalized scaling reduction

$$U = \frac{f(z)}{T^2} + d, \qquad z = \frac{x}{T} + c(t_5), \qquad T = [10g_1t_5]^{1/5}, \tag{2.17}$$

where $g_1 \neq 0$ and d are arbitrary constants, to the case $n = 2$ of (2.10), (2.11), that is,

$$K[f](K[f])_{zz} - \frac{1}{2}((K[f])_z)^2 + 2f(K[f])^2 + \frac{1}{2}(g_1 + \alpha_2)^2 = 0 \tag{2.18}$$

with

$$K[f] = f_{zz} + 3f^2 + B_1f + B_0\frac{1}{2} + g_1z, \tag{2.19}$$

where

$$\beta_1 = \frac{B_1}{T^2} - 10d, \qquad \text{and} \qquad c = -\frac{3d^2}{g_1}T^4 + \frac{dB_1}{g_1}T^2 - \frac{B_0}{2g_1} + \frac{\tilde{c}}{T}, \tag{2.20}$$

$\tilde{c}$ being an arbitrary constant.

3 The dispersive water wave case

The DWW hierarchy is a two-component hierarchy in $\mathbf{u} = (u, v)^T$ given by [6]

$$\mathbf{u}_{t_n} = \mathcal{R}^n[\mathbf{u}]\mathbf{u}_x, \qquad \mathcal{R}[\mathbf{u}] = \frac{1}{2}\begin{pmatrix} \partial_x u\partial_x^{-1} - \partial_x & 2 \\ 2v + v_x\partial_x^{-1} & u + \partial_x \end{pmatrix}, \tag{3.1}$$

where the recursion operator $\mathcal{R}$ is the quotient $\mathcal{R}[\mathbf{u}] = \mathcal{B}_2[\mathbf{u}]\mathcal{B}_1^{-1}[\mathbf{u}]$ of the two Hamiltonian operators

$$\mathcal{B}_2[\mathbf{u}] = \frac{1}{2}\begin{pmatrix} 2\partial_x & \partial_x u - \partial_x^2 \\ u\partial_x + \partial_x^2 & v\partial_x + \partial_x v \end{pmatrix}, \tag{3.2}$$

and

$$\mathcal{B}_1[\mathbf{u}] = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}. \tag{3.3}$$

The DWW hierarchy can be written

$$\mathbf{u}_{t_n} = B_1[\mathbf{u}]\mathbf{L}_{n+1}[\mathbf{u}] = B_2[\mathbf{u}]\mathbf{L}_n[\mathbf{u}], \tag{3.4}$$

where the quantities $\mathbf{L}_n[\mathbf{u}]$, defined by $\mathbf{L}_0[\mathbf{u}] = (0,2)^T$ and by the recursion relation given by the last equality in equation (3.4),

$$\mathbf{L}_0[\mathbf{u}] = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \qquad \mathbf{L}_1[\mathbf{u}] = \begin{pmatrix} v \\ u \end{pmatrix}, \qquad \mathbf{L}_2[\mathbf{u}] = \frac{1}{2}\begin{pmatrix} 2uv + v_x \\ 2v + u^2 - u_x \end{pmatrix}, \qquad \ldots \tag{3.5}$$

are the variational derivatives of a corresponding sequence of Hamiltonian densities. We do not need the expressions for these Hamiltonian densities here. Neither do we need the third Hamiltonian operator of the (tri-Hamiltonian) DWW hierarchy.

Here we show that a suitable generalization of this hierarchy can be used to derive a fourth Painlevé hierarchy via a generalized scaling reduction. We begin by recalling the following Lemma [1].

Lemma 2. The change of variables $\tilde{\mathbf{u}} = (u + C, v)^T$, where C is an arbitrary constant, in $\mathcal{R}^n[\tilde{\mathbf{u}}]\tilde{\mathbf{u}}_x$, yields

$$\mathcal{R}^n[\tilde{\mathbf{u}}]\tilde{\mathbf{u}}_x = \sum_{j=0}^{n} \alpha_{n,j} C^{n-j} \mathcal{R}^j[\mathbf{u}]\mathbf{u}_x, \tag{3.6}$$

where the coefficients $\alpha_{n,j}$ are determined recursively by

$$\alpha_{n,n} = 1, \tag{3.7}$$

$$\alpha_{n,j} = \frac{1}{2}\alpha_{n-1,j} + \alpha_{n-1,j-1}, \qquad j = 1, \ldots, n-1, \tag{3.8}$$

$$\alpha_{n,0} = \frac{1}{2}\left(\frac{n+1}{n}\right)\alpha_{n-1,0} \tag{3.9}$$

and where $\alpha_{0,0} = 1$.

Proposition 2. There exists a choice of coefficient funtions $\gamma_i(t_n)$ and of the function $c(t_n)$ such that the substitution

$$u = \frac{f(z)}{\left[\frac{1}{2}(n+1)g_n t_n\right]^{1/(n+1)}} + d, \qquad v = \frac{g(z)}{\left[\frac{1}{2}(n+1)g_n t_n\right]^{2/(n+1)}}, \qquad z = \frac{x}{\left[\frac{1}{2}(n+1)g_n t_n\right]^{1/(n+1)}} + c(t_n), \tag{3.10}$$

where $g_n \neq 0$ and d are arbitrary constants, into the hierarchy

$$\mathbf{u}_{t_n} = \mathcal{R}^n[\mathbf{u}]\mathbf{u}_x + \sum_{i=1}^{n-1} \gamma_i(t_n)\mathcal{R}^i[\mathbf{u}]\mathbf{u}_x \tag{3.11}$$

yields the fourth Painlevé hierarchy in $\mathbf{f} = (f,g)^T$,

$$0 = 2K + fL + g_n - 2\alpha_n - L_z, \tag{3.12}$$

$$0 = \left(K + \frac{1}{2}g_n - \alpha_n\right)^2 - \frac{1}{4}\beta_n^2 - gL^2 - K_z L. \tag{3.13}$$

where α_n and β_n are arbitrary constants and K and L are the components of $\mathbf{K}[\mathbf{f}]$, $\mathbf{K} = (K, L)^T$, this last being given by

$$\mathbf{K}[\mathbf{f}] = \mathbf{L}_n[\mathbf{f}] + \sum_{i=0}^{n-1} B_i \mathbf{L}_i[\mathbf{f}] + g_n \begin{pmatrix} 0 \\ z \end{pmatrix}, \tag{3.14}$$

wherein the coefficients B_i are arbitrary constants and $\mathbf{L}_j[\mathbf{f}]$ denotes the variational derivatives of the Hamiltonian densities of the DWW hierarchy with dependent variables f and g and independent variable z.

Proof. Using Lemma 2 we see that substituting (3.10) in (3.11) gives

$$\sum_{k=0}^{n} \mathbf{\Gamma}_k \mathcal{R}^k[\mathbf{f}]\mathbf{f}_z + \frac{1}{2} g_n \mathbf{T}_{n+2} \begin{pmatrix} (zf)_z \\ 2g + zg_z \end{pmatrix} \equiv \sum_{j=0}^{n} \alpha_{n,j} d^{n-j} \mathbf{T}_{j+2} \mathcal{R}^j[\mathbf{f}]\mathbf{f}_z$$
$$+ \sum_{i=1}^{n-1} \left(\gamma_i \sum_{j=0}^{i} \alpha_{i,j} d^{i-j} \mathbf{T}_{j+2} \mathcal{R}^j[\mathbf{f}]\mathbf{f}_z \right) - \mathbf{T}_1 \mathbf{f}_z c_{t_n} - \frac{1}{2} g_n \mathbf{T}_{n+2} \mathbf{f}_z c + \frac{1}{2} g_n \mathbf{T}_{n+2} \begin{pmatrix} (zf)_z \\ 2g + zg_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.15}$$

where

$$\mathbf{T}_j = \begin{pmatrix} 1/T^j & 0 \\ 0 & 1/T^{j+1} \end{pmatrix}, \qquad T = [(n+1)g_n t_n/2]^{1/(n+1)}, \tag{3.16}$$

and $\mathcal{R}[\mathbf{f}]$ is obtained from $\mathcal{R}[\mathbf{u}]$ by replacing $\mathbf{u}$ by $\mathbf{f}$ and ∂_x by ∂_z. We recall that each $\alpha_{i,i} = 1$, and so in particular $\mathbf{\Gamma}_n = \mathbf{T}_{n+2}$.

We solve the equations

$$\mathbf{\Gamma}_k = B_k \mathbf{T}_{n+2}, \qquad k = n-1, \dots, 1 \tag{3.17}$$

recursively for the coefficients γ_k and the equation

$$\mathbf{\Gamma}_0 = B_0 \mathbf{T}_{n+2} \tag{3.18}$$

for c, where all B_k are constants. The resulting ODE can be written

$$\mathcal{B}_2[\mathbf{f}]\mathbf{K}[\mathbf{f}] = 0 \tag{3.19}$$

where $\mathcal{B}_2[\mathbf{f}]$ is obtained from $\mathcal{B}_2[\mathbf{u}]$ by replacing $\mathbf{u}$ by $\mathbf{f}$ and ∂_x by ∂_z, and $\mathbf{K}[\mathbf{f}]$ is as given in (3.14). This last system integrates to (3.12), (3.13), where α_n and β_n are arbitrary constants of integration.

Remark 2. Without loss of generality, we may set, using a shift on z, $B_0 = 0$ in (3.12), (3.13), (3.14). This then gives the version of the fourth Painlevé hierarchy defined in [7, 8]. We remark that the fourth Painlevé hierarchy was originally given in [9]; the case with all $B_k = 0$ can be obtained from the non-generalized scaling reduction ($d = 0$ and $c = 0$) of the standard DWW hierarchy (3.1).

Example 2. The second nontrivial dispersive water wave flow

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_2} = \frac{1}{4} \begin{pmatrix} u_{xx} - 3uu_x + u^3 + 6uv \\ v_{xx} + 3v^2 + 3uv_x + 3u^2 v \end{pmatrix}_x + \frac{1}{2}\gamma_1(t_2) \begin{pmatrix} 2v + u^2 - u_x \\ 2uv + v_x \end{pmatrix}_x \tag{3.20}$$

admits the generalized scaling reduction

$$u = \frac{f(z)}{T} + d, \qquad v = \frac{g(z)}{T^2}, \qquad z = \frac{x}{T} + c(t_2), \qquad T = \left[\frac{3}{2} g_2 t_2\right]^{1/3}, \tag{3.21}$$

where $g_2 \neq 0$ and d are arbitrary constants, to the case $n = 2$ of (3.12), (3.13), (3.14), that is,

$$0 = 2K + fL + g_2 - 2\alpha_2 - L_z, \tag{3.22}$$

$$0 = \left(K + \frac{1}{2}g_2 - \alpha_2\right)^2 - \frac{1}{4}\beta_2^2 - gL^2 - K_z L, \tag{3.23}$$

with

$$\begin{pmatrix} K \\ L \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2fg + g_z \\ 2g + f^2 - f_z \end{pmatrix} + B_1 \begin{pmatrix} g \\ f \end{pmatrix} + B_0 \begin{pmatrix} 0 \\ 2 \end{pmatrix} + g_2 \begin{pmatrix} 0 \\ z \end{pmatrix}, \tag{3.24}$$

where

$$\gamma_1 = \frac{B_1}{T} - \frac{3}{2}d, \quad \text{and} \quad c = -\frac{d^2}{2g_2}T^2 + \frac{dB_1}{g_2}T - \frac{2B_0}{g_2} + \frac{\tilde{c}}{T}, \tag{3.25}$$

$\tilde{c}$ being an arbitrary constant.

4 The Burgers case

The Burgers hierarchy is given by [10, 11, 12, 13]

$$U_{t_{n+1}} = \mathcal{R}^n[U]U_x, \qquad \mathcal{R}[U] = \partial_x \left(\partial_x + \frac{1}{2}U\right)\partial_x^{-1}, \tag{4.1}$$

or alternatively

$$U_{t_{n+1}} = \partial_x \mathcal{L}_n[U] = \partial_x \mathcal{T}^n[U]U, \qquad \mathcal{T}[U] = \partial_x + \frac{1}{2}U. \tag{4.2}$$

We find that our construction of generalized scaling reductions can also be realized for an extended version of this hierarchy, resulting in a hierarchy of linearizable ODEs. We begin by recalling the following Lemma [1].

Lemma 3. The change of variables $\tilde{U} = U + C$, where C is an arbitrary constant, in $\mathcal{L}_n[\tilde{U}]$, yields

$$\mathcal{L}_n[\tilde{U}] = \sum_{j=-1}^{n} \begin{pmatrix} n+1 \\ j+1 \end{pmatrix} \left(\frac{1}{2}C\right)^{n-j} \mathcal{L}_j[U] \tag{4.3}$$

where we define $\mathcal{L}_{-1}[U] = 2$.

Proposition 3. There exists a choice of coefficient funtions $\beta_i(t_{n+1})$ and of the function $c(t_{n+1})$ such that the substitution

$$U = \frac{f(z)}{[(n+1)g_{n-1}t_{n+1}]^{1/(n+1)}} + d, \qquad z = \frac{x}{[(n+1)g_{n-1}t_{n+1}]^{1/(n+1)}} + c(t_{n+1}), \tag{4.4}$$

where $g_{n-1} \neq 0$ and d are arbitrary constants, into the hierarchy

$$U_{t_{n+1}} = \mathcal{R}^n[U]U_x + \sum_{i=1}^{n-1} \beta_i(t_{n+1})\mathcal{R}^i[U]U_x \tag{4.5}$$

yields the hierarchy of ODEs

$$\mathcal{L}_n[f] + \sum_{i=-1}^{n-1} B_i \mathcal{L}_i[f] + g_{n-1} z f = 0 \tag{4.6}$$

where $\mathcal{L}_n[f]$ is defined as above but with dependent variable f and independent variable z, and where the coefficients B_i are arbitrary constants.

Proof. Using Lemma 3 we see that substituting (4.4) in (4.5) gives

$$\sum_{k=0}^{n}\gamma_k\mathcal{R}^k[f]f_z+\frac{g_{n-1}}{T^{n+2}}(zf)_z\equiv\sum_{j=0}^{n}\left(\begin{array}{c}n+1\\ j+1\end{array}\right)\left(\frac{1}{2}d\right)^{n-j}\frac{1}{T^{j+2}}\mathcal{R}^j[f]f_z$$
$$+\sum_{i=1}^{n-1}\left(\beta_i\sum_{j=0}^{i}\left(\begin{array}{c}i+1\\ j+1\end{array}\right)\left(\frac{1}{2}d\right)^{i-j}\frac{1}{T^{j+2}}\mathcal{R}^j[f]f_z\right)-\frac{1}{T}f_zc_{t_{n+1}}-\frac{g_{n-1}}{T^{n+2}}f_zc+\frac{g_{n-1}}{T^{n+2}}(zf)_z=0 \quad (4.7)$$

where $T=[(n+1)g_{n-1}t_{n+1}]^{1/(n+1)}$, where we have used the fact that L_{-1} is constant, and where clearly $\gamma_n=1/T^{n+2}$. We solve the equations

$$\gamma_k=B_k/T^{n+2},\qquad k=n-1,\ldots,1 \quad (4.8)$$

recursively for the coefficients β_k and the equation

$$\gamma_0=B_0/T^{n+2} \quad (4.9)$$

for c, where all B_k are constants. Integrating the resulting ODE then yields (4.6), where we include a constant of integration as the term $B_{-1}L_{-1}[f]=2B_{-1}$.

Remark 3. Without loss of generality, we may set, using a shift on z, $B_0=0$ in (4.6).

Proposition 4. The hierarchy (4.6) is linearizable using the Cole-Hopf transformation $f=2\varphi_z/\varphi$ [14, 15, 16] onto the hierarchy of ODEs

$$\partial_z^{n+1}\varphi+\sum_{i=-1}^{n-1}B_i\partial_z^{i+1}\varphi+g_{n-1}z\varphi_z=0. \quad (4.10)$$

Proof. This follows immediately from the analogous result in [11] for the Burgers hierarchy.

Remark 4. The general solution of (4.10) can be obtained in terms of an everywhere-convergent power series.

Remark 5. In the special case of the standard Burgers flows (all $\beta_i=0$), the non-generalized scaling reduction ($d=0$ and $c=0$) to a linearizable ODE (all $B_i=0$ for $i\geq 0$ in (4.6)) has been considered in [17].

Example 3. The second nontrivial member of the Burgers hierarchy,

$$U_{t_3}=\left(U_{xx}+\frac{3}{2}UU_x+\frac{1}{4}U^3\right)_x+\beta_1(t_3)\left(U_x+\frac{1}{2}U^2\right)_x, \quad (4.11)$$

admits the generalized scaling reduction

$$U=\frac{f(z)}{T}+d,\qquad z=\frac{x}{T}+c(t_3),\qquad T=[3g_1t_3]^{1/3}, \quad (4.12)$$

where $g_1\neq 0$ and d are arbitrary constants, to the case $n=2$ of (4.6), that is,

$$f_{zz}+\frac{3}{2}ff_z+\frac{1}{4}f^3+B_1(f_z+\frac{1}{2}f^2)+B_0f+2B_{-1}+g_1zf=0, \quad (4.13)$$

where

$$\beta_1 = \frac{B_1}{T} - \frac{3}{2}d, \qquad \text{and} \qquad c = -\frac{d^2}{4g_1}T^2 + \frac{B_1 d}{2g_1}T - \frac{B_0}{g_1} + \frac{\tilde{c}}{T}, \tag{4.14}$$

$\tilde{c}$ being an arbitrary constant.

We note that equation (4.13) is linearizable via the Cole-Hopf transformation $f = 2\varphi_z/\varphi$ onto the ODE

$$\varphi_{zzz} + B_1\varphi_{zz} + B_0\varphi_z + B_{-1}\varphi + g_1 z\varphi_z = 0. \tag{4.15}$$

5 Conclusions

We have given new derivations of two Painlevé hierarchies, as well as a derivation of a hierarchy of linearizable ODEs, by considering generalized scaling reductions of the Korteweg-de Vries, dispersive water wave and Burgers hierarchies augmented by lower order flows with coefficients functions of the flow time. The ODE hierarchies obtained include lower-weight terms. To the best of our knowledge, generalized scaling reductions of integrable hierarchies have not previously been considered in the literature. Our results complement our earlier work on accelerating-wave type reductions of integrable hierarchies. In future papers we will consider the application of our approach to other integrable hierarchies, for example to the Boussinesq hierarchy.

Finding the associated linear problems, or Lax pairs, for the hierarchies of the Painlevé equations is an interesting and challenging problem. In 2001, linear problems for P_{II} and P_{IV} hierarchies were obtained from the generalized non-isospectral dispersive water wave hierarchy in $2+1$ dimensions [9]. In [8], the relation between the linear problems for the P_{II} and P_{IV} hierarchies obtained in [9] and other linear problems was given, and it was shown that there exists gauge transformations which map the linear problems for the P_{II} and P_{IV} hierarchies onto new linear problems such that their first members are the linear problems of P_{II} and P_{IV} given by Jimbo and Miwa [18]. In [19], Kudryashov found a new hierarchy of ODEs (which is a generalization of the P_{II} hierarchy) and associated linear problems by using the generalization of the isomonodromic linear problems for P_{II}. In [20], new hierarchies of nonlinear ODEs which contain the Painlevé equations as a special cases were given. In [21], by expanding the Jimbo-Miwa isomonodromy problems of P_I, P_{II}, P_{III} and P_{IV} in powers of the spectral variable λ, isomonodromic linear problems for the hierarchies of P_I, P_{II}, P_{III} and P_{IV} were obtained. Moreover, some special solutions of the hierarchies of P_{II}, P_{III} and P_{IV} were given.

Once the members of the hierarchy are presented as the compatibility conditions of the isomonodromic linear problems, these problems can be used to solve the Cauchy problems of the members of the hierarchy by the Inverse Monodromy Transform (IMT). The Cauchy problem for the second member of a P_{IV} hierarchy was studied in [22] by using the Lax pair introduced in [8]. One can also obtain Schlesinger transformations and special solutions of Painlevé hierarchies by using the isomonodromy problem. Schlesinger transformations for the second and fourth Painlevé hierarchies were studied in [23]. Lax pairs, Cauchy problems, special solutions and Schlesinger transformations for Painlevé hierarchies will be the subject of forthcoming articles.

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