Fusion systems seminar bonus:
Characteristic bisets and Frobenius reciprocity

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Goals:

(i) Give intrinsic condition on a biset Ω that implies $\mathcal{F}_\Omega = \mathcal{F}^\Omega$.
(ii) Explore implications of Frobenius reciprocity for the structure of Ω^n.

We finish our discussion of characteristic bisets by giving an intrinsic condition, called *Frobenius reciprocity*, that is necessary and sufficient to guarantee $\mathcal{F}_\Omega = \mathcal{F}^\Omega$. In particular, we do not need to reference an external fusion system, as we did in the previous lecture. This will involve placing a condition on various S^3-actions of Ω^2, which we generalize to S^{n+1}-actions on Ω^n. The first two sections come from the work of Ragnarsson-Stancu, while the last two are more speculative observations.

Return to motivating example

We consider again the situation where S is a Sylow p-subgroup of the finite group G, which we regard as the (S, S)-biset ${}_SG_S$. Previously we considered the relationship between the G-fusion on S and ${}_SG_S$, considered primarily in terms of point-stabilizers and the symmetry of ${}_SG_S$ induced by inversion. The one thing we did not make use of was the multiplicative structure of G,

$$\mu : G \times G \to G : (g_1, g_2) \mapsto g_1g_2,$$

and the induced properties of ${}_SG_S$. We correct this omission now.

While there are two natural free S-actions on G—left and right multiplication—there are three natural free S-actions on $G \times G$ that are compatible with the multiplication μ: Left multiplication on the left factor, right multiplication on the right factor, and the combined diagonal action of right action on the left factor with left action on the right. If we denote these three actions by ℓ, ρ, and δ, respectively, we have

$$\begin{aligned} \ell_a(g_1, g_2) &= (ag_1, g_2), \\ \rho_a(g_1, g_2) &= (g_1, g_2a^{-1}), \text{ and} \\ \delta_a(g_1, g_2) &= (g_1a^{-1}, ag_2) \end{aligned}$$

for all $a \in S$, $g_1, g_2 \in G$. We write the right multiplications with an inverse so that we have $\ell_{ab} = \ell_a\ell_b$, $\rho_{ab} = \rho_a\rho_b$, and $\delta_{ab} = \delta_a\delta_b$; we are secretly thinking of all three actions as being covariant (i.e., on the left) and transforming some into contravariant actions by composition with the inverse. "Compatible" means that μ is S-equivariant when $\ell^{G\times G}$ is paired with ℓ^G, $\rho^{G\times G}$ is paired with ρ^G, and δ is paired with the trivial S-action on G:

$$\begin{aligned} \mu\ell_a(g_1, g_2) &= \ell_a\mu(g_1, g_2), \\ \mu\rho_a(g_1, g_2) &= \rho_a\mu(g_1, g_2), \text{ and} \\ \mu\delta_a(g_1, g_2) &= \mu(g_1, g_2). \end{aligned}$$

Since ℓ and ρ on $G\times$ correspond with nontrivial actions on G, while the effect of δ is killed by μ, we choose to view ℓ and ρ as left actions on $G \times G$ and δ as a right action.

Thus $G \times G$ becomes a $(S \times S, S)$-biset, where

$$(s_1, s_2) \cdot (g_1, g_2) \cdot t = (s_1 g_1 t, s_2 g_2 t)$$

for all $(s_1, s_2) \in S \times S$, $t \in S$, and $(g_1, g_2) \in G \times G$. In order for multiplication to be equivariant with respect to this action, we must precompose with inversion on the second factor, obtaining

$$\widetilde{\mu} : G \times G \to G : (g_1, g_2) \mapsto g_1 g_2^{-1}.$$

Then if we view G as an $(S \times S, S)$-biset with left $S \times S$-action given by left and right (inverse) multiplication and right S-action trivial,

$$(s_1, s_2) \cdot g \cdot t = s_1 g s_2^{-1},$$

we have $\widetilde{\mu} : G \times G \to G$ is a morphism of $(S \times S, S)$-bisets:

$$(s_1, s_2) \cdot \widetilde{\mu}(g_1, g_2) \cdot t = \widetilde{\mu}((s_1, s_2) \cdot (g_1, g_2) \cdot t)$$

Now, $\widetilde{\mu}$ is not an isomorphism of $(S \times S, S)$-bisets, as it is surjective but not injective. We could correct for this by noting that $\widetilde{\mu}(g_0, ?) : G \to G$ and $\widetilde{\mu}(?, g_0) : G \to G$ is necessarily bijective for all $g_0 \in G$.

A nice way of encoding this by noting that $\widetilde{\mu} \times \mathrm{pr}_2 : G \times G \to G \times G$ is a bijection. Moreover, if we view the source as an $(S \times S, S)$-biset with action $*$,

$$(s_1, s_2) * (g_1, g_2) * t = (s_1 g_1 t, s_2 g_2 t),$$

and the target as having the action $\odot$,

$$(s_1, s_2) \odot (g_1, g_2) \odot t = (s_1 g_1 s_2^{-1}, s_2 g_2 t),$$

then $\widetilde{\mu} \times \mathrm{pr}_2$ is an isomorphism of $(S \times S, S)$-bisets $(G \times G, *) \cong (G \times G, \odot)$. This is the key property we wish to generalize for arbitrary (S, S)-bisets.

Frobenius reciprocity of bisets

Throughout this section, let Ω be a bifree symmetric (S, S)-biset containing $[\mathrm{id}, S]$.

We consider two $(S \times S, S)$-structures, denoted $*$ and $\odot$, on $\Omega \times \Omega$, lifted from the motivating example $\Omega = {}_S G_S$. Define

$$\begin{aligned} (s_1, s_2) * (\omega_1, \omega_2) * t &= (s_1 \cdot \omega_1 \cdot t, s_2 \cdot \omega_2 \cdot t) \text{ and} \\ (s_1, s_2) \odot (\omega_1, \omega_2) \odot t &= (s_1 \cdot \omega_1 \cdot s_2^{-1}, s_2 \cdot \omega_2 \cdot t). \end{aligned}$$

Definition 1. Ω *satisfies Frobenius reciprocity*, or is an *Frobenius reciprocity biset* if

$$(\Omega \times \Omega, *) \cong (\Omega \times \Omega, \odot).$$

Our goal is to show that if Ω is a Frobenius reciprocity biset, then the two fusion systems associated to Ω are equal. Recall these definitions now:

Definition 2. Let Ω be a bifree symmetric (S, S)-biset containing $[\mathrm{id}_S, S]$.

(i) $\mathcal{F}_\Omega$ is the fusion presystem on S with

$$\begin{aligned} \mathcal{F}_\Omega(P, Q) &= \mathrm{Hom}_\Omega(P, Q) \\ &= \{\varphi \in \mathrm{Hom}(P, Q) \big| \exists\, \omega \in \Omega \text{ s.th. } P \leq S_\omega \text{ and } c_\omega|_P = \varphi\} \\ &= \{\varphi \in \mathrm{Hom}(P, Q) \big| \exists\, \omega \in \Omega \text{ s.th. } \varphi(a) \cdot \omega = \omega \cdot a \;\forall\, a \in P\} \\ &= \{\varphi \in \mathrm{Hom}(P, Q) \big| \Omega^{(\varphi, P)} \neq \emptyset\}. \end{aligned}$$

(ii) $\mathcal{F}^\Omega$ is the fusion system on S obtained by identifying S with $\widehat{S} \leq \widehat{G} := \mathrm{Aut}({}_1\Omega_S)$. Thus

$$\begin{aligned}\mathcal{F}^\Omega(P,Q) &= \{\varphi \in \mathrm{Hom}(P,Q) \big| {}_P\Omega_S \cong {}^{\varphi}_{P}\Omega_S\} \\ &= \{\varphi \in \mathrm{Hom}(P,Q) \big| \exists\, \sigma \in \widehat{G} \text{ s.th. } \sigma(a\cdot\omega) = \varphi(a)\cdot\sigma(\omega)\ \forall\, a \in P, \omega \in \Omega\}.\end{aligned}$$

Here, *fusion presystem* simply means that $\mathcal{F}_\Omega$ need not be a category, as described in the previous talk. However, we at least have $\mathrm{id}_P \in \mathcal{F}_\Omega(P,P)$ for all $P \leq S$ because $[\mathrm{id}_S, S] \subseteq \Omega$. In fact, we can use this inclusion to reproduce in a more straightforward manner our result that $\mathcal{F}^\Omega \subseteq \mathcal{F}_\Omega$.

As $[\mathrm{id}_S, S] \subseteq \Omega$, there exists an element of Ω whose (S,S)-stabilizer is (id_S, S). Pick one such element and denote it 1_Ω.

Proposition 3. *Let Ω and 1_Ω be as above. Then $\mathcal{F}^\Omega \subseteq \mathcal{F}_\Omega$.*

Proof. Suppose that $\varphi \in \mathcal{F}^\Omega(P,S)$ is realized by $\sigma \in \widehat{G}$, so that

$$\sigma(a\cdot\omega) = \varphi(a)\cdot\sigma(\omega)$$

for all $a \in P$ and $\omega \in \Omega$. Consider the element $\sigma(1_\Omega)$. I claim that the point-stabilizer of $\sigma(1_\Omega)$ contains (φ, P), so that this element will realize $\varphi \in \mathcal{F}_\Omega(P,S)$. For all $a \in P$, we have

$$\varphi(a)\cdot\sigma(1_\Omega) = \sigma(a\cdot 1_\Omega) = \sigma(1_\Omega\cdot a) = \sigma(1_\Omega)\cdot a,$$

and we're done. □

We now turn to showing that if Ω is a Frobenius reciprocity biset, we also have the reverse inclusion $\mathcal{F}_\Omega \subseteq \mathcal{F}^\Omega$.

To do this, we first described the fixed point orders of the two $(S\times S, S)$-biset structures on $\Omega\times\Omega$. Note that both the left $S\times S$ and right S-actions are free for $*$ and $\odot$, so the stabilizer of a point (ω_1,ω_2) is a twisted diagonal subgroup of the form (Φ, P) for $P \leq S$ and $\Phi \in \mathrm{Inj}(P, S\times S)$. Write $\Phi = \varphi\times\psi$ for $\varphi,\psi \in \mathrm{Hom}(A,S)$. We also adopt the notation $\Omega \overset{*}{\times} \Omega$ for $(\Omega\times\Omega, *)$, and $\Omega \overset{\odot}{\times} \Omega$ for $(\Omega\times\Omega,\odot)$. Note that both $\Omega \overset{*}{\times} \Omega$ and $\Omega \overset{\odot}{\times} \Omega$ are bifree as $(S\times 1, S)$ and $(1\times S, S)$-bisets, from which it follows that φ and ψ must both be injective as well.

Proposition 4. *Fix $(\varphi\times\psi, P) \leq (S\times S)\times S$ with $P \leq S$ and $\varphi,\psi \in \mathrm{Inj}(P,S)$.*

(i) $\Omega \overset{*}{\times} \Omega^{(\varphi\times\psi,P)} = \Omega^{(\varphi,P)} \times \Omega^{(\psi,P)}$.

(ii) $\Omega \overset{\odot}{\times} \Omega^{(\varphi\times\psi,P)} = \Omega^{(\varphi\psi^{-1},\psi P)} \times \Omega^{(\psi,P)}$.

Proof.

(i) Suppose that $(\omega_1,\omega_2) \in \Omega \overset{*}{\times} \Omega^{(\varphi\times\psi,P)}$ so that for all $a \in P$,

$$(\varphi(a)\cdot\omega_1, \psi(a)\cdot\omega_2) = (\omega_1\cdot a, \omega_2\cdot a),$$

which says that $\omega_1 \in \Omega^{(\varphi,P)}$ and $\omega_2 \in \Omega^{(\psi,P)}$. The reverse inclusion is obvious.

(ii) Suppose that $(\omega_1,\omega_2) \in \Omega \overset{\odot}{\times} \Omega^{(\varphi\times\psi,P)}$ so that for all $a \in P$,

$$(\varphi(a)\cdot\omega_1\cdot\psi(a)^{-1}, \psi(a)\cdot\omega_2) = (\omega_1, \omega_2\cdot a).$$

The second coordinate implies that $\omega_2 \in \Omega^{(\psi,P)}$, and the first can be rewritten

$$\varphi(a)\cdot\omega_1 = \omega_1\cdot\psi(a).$$

Since ψ is injective, this is equivalent to saying that $\omega_1 \in (\varphi\psi^{-1}, \psi P)$. Again, the reverse inclusion follows from undoing these steps.

$\square$

Corollary 5. *If Ω satisfies Frobenius reciprocity, then for all $P \leq S$ and $\varphi, \psi \in \mathrm{Inj}(P, S)$, we have*

$$\left|\Omega^{(\varphi,P)}\right| \cdot \left|\Omega^{(\psi,P)}\right| = \left|\Omega^{(\varphi\psi^{-1},\psi P)}\right| \cdot \left|\Omega^{(\psi,P)}\right|.$$

In particular, if $\Omega^{(\psi,P)} \neq \emptyset$, then

$$\left|\Omega^{(\varphi,P)}\right| = \left|\Omega^{(\varphi\psi^{-1},\psi P)}\right|.$$

Proof. Ω is a Frobenius reciprocity biset implies $\Omega \overset{*}{\times} \Omega \cong \Omega \overset{\odot}{\times} \Omega$ as $(S \times S, S)$-bisets, so in particular there is an equality of fixed-point orders for all subgroups of $(S \times S) \times S$. The result is then immediate from the previous Proposition. $\square$

Theorem 6. *If Ω is a symmetric bifree Frobenius reciprocity biset that contains $[\mathrm{id}_S, S]$, then $\mathcal{F}_\Omega = \mathcal{F}^\Omega$.*

Proof. From the previous comments, we need only $\mathcal{F}_\Omega \subseteq \mathcal{F}^\Omega$. Suppose that $\psi \in \mathcal{F}_\Omega(P, S)$, so that $\Omega^{(\psi,P)} \neq \emptyset$; we must prove the (P, S)-bisets ${}_P\Omega_S$ and ${}^{\psi}_{P}\Omega_S$ are isomorphic. We will do this by showing that for all $Q \leq P \times S$, we have an equality of fixed-point orders

$$\left|{}_P\Omega_S^Q\right| = \left|{}^{\psi}_{P}\Omega_S^Q\right|.$$

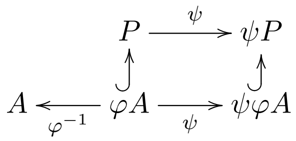
Since every point-stabilizer of Ω is a twisted diagonal subgroup, if Q is not of this form then both fixed point sets are empty. Thus we may assume $Q = (\varphi, A)$ for $A \leq S$ and $\varphi \in \mathrm{Inj}(A, P)$. We then have

$${}^{\psi}_{P}\Omega_S^{(\varphi,A)} = \left\{\omega \in \Omega \middle| \psi(\varphi(a)) \cdot \omega = \omega \cdot a\right\} = \Omega^{(\psi\varphi,A)}.$$

Since ${}_P\Omega_S^{(\varphi,A)} = \Omega^{(\varphi,A)}$, we must show

$$\left|\Omega^{(\psi\varphi,A)}\right| = \left|\Omega^{(\varphi,A)}\right|$$

for all such (φ, A). Consider:



where we denote the restriction of ψ to φA by ψ. Since $\Omega^{(\psi,P)} \subseteq \Omega^{(\psi,\varphi A)}$, we have the latter is nonempty, so we may apply the previous proposition. We then have

$$\left|\Omega^{(\varphi,A)}\right| = \left|\Omega^{(\varphi^{-1},\varphi A)}\right| = \left|\Omega^{(\varphi^{-1}\psi^{-1},\psi\varphi A)}\right| = \left|\Omega^{((\psi\varphi)^{-1},\psi\varphi A)}\right| = \left|\Omega^{(\psi\varphi,A)}\right|,$$

where the first and last equalities use the symmetry of Ω, the second uses the previous Corollary, and the third is obvious. This complete the proof.

$\square$

Note that since we already know that $\mathcal{F}^\Omega$ is a category (and in fact a fusion system), this Theorem in particular implies that $\mathcal{F}_\Omega$ is as well.

If Ω if a Frobenius reciprocity biset, we write $\mathcal{F}$ for the common fusion system $\mathcal{F}_\Omega = \mathcal{F}^\Omega$.

Theorem 7. *If Ω is a Frobenius reciprocity biset with fusion system $\mathcal{F}$, then Ω is $\mathcal{F}$-generated and $\mathcal{F}$-stable.*

Proof. For an arbitrary (S,S)-biset Ξ and fusion system $\mathcal{G}$ on S, Ξ is $\mathcal{G}$-generated iff $\mathcal{F}_\Xi \subseteq \mathcal{G}$, and Ξ is is $\mathcal{G}$-stable iff $\mathcal{G} \subseteq \mathcal{F}^\Xi$. The result is now obvious. □

Finally, show that Frobenius reciprocity is truly an intrinsic reformulation of $\mathcal{F}$-generation and $\mathcal{F}$-stability.

Theorem 8. *Let $\mathcal{F}$ be a fusion system on S and Ω an (S,S)-biset. If Ω is $\mathcal{F}$-generated and $\mathcal{F}$-stable, then Ω is a Frobenius reciprocity biset.*

Proof. Recall that the condition on Ω implies that $\mathcal{F}_\Omega = \mathcal{F} = \mathcal{F}^\Omega$. We must show that $\Omega \overset{*}{\times} \Omega \cong \Omega \overset{\odot}{\times} \Omega$ as $(S\times S, S)$-bisets. As Ω is bifree, Proposition 4 and its Corollary imply that we must show for all $P \le S$ and $\varphi, \psi \in \mathrm{Inj}(P,S)$ we have

$$\left|\Omega^{(\varphi,P)}\right| \cdot \left|\Omega^{(\psi,P)}\right| = \left|\Omega^{(\varphi\psi^{-1},\psi P)}\right| \cdot \left|\Omega^{(\psi,P)}\right|.$$

We may assume that $\Omega^{(\psi,P)} \neq \emptyset$, or equivalently $\psi \in \mathcal{F}_\Omega$, as otherwise the equality is obvious. So we must show

$$\left|\Omega^{(\varphi,P)}\right| = \left|\Omega^{(\varphi\psi^{-1},\psi P)}\right| \tag{1}$$

whenever $\psi \in \mathcal{F}_\Omega$ and $\varphi \in \mathrm{Inj}(P,S)$. Suppose that $\varphi \notin \mathcal{F}_\Omega$. Then $\varphi = \varphi\psi^{-1} \circ \psi$, so that (since $\mathcal{F}_\Omega = \mathcal{F}^\Omega$, which is a fusion system and so is closed under taking composites and inverses) we conclude that $\varphi\psi^{-1}$ must not be in $\mathcal{F}_\Omega$ either. Thus we are reduced to proving Equation (1) in the case when both φ and ψ are morphisms of $\mathcal{F}$. We compute:

$$\left|\Omega^{(\varphi\psi^{-1},\psi P)}\right| = \left|\Omega^{(\iota,\psi P)}\right| = \left|\Omega^{(\psi^{-1},\psi P)}\right| = \left|\Omega^{(\psi,P)}\right| = \left|\Omega^{(\iota,P)}\right| = \left|\Omega^{(\varphi,P)}\right|,$$

where the middle equality is due to the symmetry of Ω and all others its $\mathcal{F}$-stability. □

Summarizing these results:

Theorem 9. *Let S be a finite p-group and Ω a finite (S,S)-biset.*

- *If Ω is a Frobenius biset, then $\mathcal{F}_\Omega = \mathcal{F}^\Omega =: \mathcal{F}$, and Ω is $\mathcal{F}$-generated and $\mathcal{F}$-stable.*
- *If $\mathcal{F}$ is a fusion system on S such that Ω is $\mathcal{F}$-generated and $\mathcal{F}$-stable, then $\mathcal{F} = \mathcal{F}_\Omega = \mathcal{F}^\Omega$, and Ω is a Frobenius biset.*

Frobenius pairings

Let us again return to the motivation for our notion of Frobenius reciprocity: The map $\widetilde{\mu} : G \times G \to G : (g,h) \mapsto gh^{-1}$. Recall that when paired with the projection map pr_2, we obtain our Frobenius isomorphism of $(S\times S, S)$-bisets

$$\widetilde{\mu} \times \mathrm{pr}_2 : G \overset{*}{\times} G \cong G \overset{\odot}{\times} G : (g,h) \mapsto (gh^{-1}, h).$$

$\widetilde{\mu}$ has the extremely nice property of being *nondegenerate*, which is to say that both

$$\widetilde{\mu}(g_0, ?) \qquad \text{and} \qquad \widetilde{\mu}(?, g_0)$$

are bijections $G \to G$ (respecting the left S-set structure) for all $g \in G$.

Definition 10. For a Frobenius biset Ω and isomorphism

$$F = (F_1, F_2) : \Omega \overset{*}{\times} \Omega \cong \Omega \overset{\odot}{\times} \Omega,$$

we say F_1 is a *Frobenius pairing* for Ω. If the maps $F_1(\omega_0, ?), F_1(?, \omega_0) : \Omega \to \Omega$ are bijections for all $\omega_0 \in \Omega$, we say that the Frobenius pairing is *nondegenerate.*

Nondegenerate Frobenius pairings are particularly nice, as the following shows:

Proposition 11. *Suppose that* F_1 *is a nondegenerate Frobenius pairing for* Ω*. Fix an element* 1_Ω *with point-stabilizer* (id, S)*.*

(i) *For* $\xi \in \Omega$*, define*

$$\ell_\xi : \Omega \to \Omega : \omega \mapsto F_1(\xi, F_1(1_\Omega, \omega)).$$

Then $\ell_\xi \in \widehat{G} = \mathrm{Aut}({}_1\Omega_S)$ *and* ℓ_ξ *induces an isomorphism* ${}_{S_\xi}\Omega_S \to {}^{c_\xi}_{S_\xi}\Omega_S$*.*

(ii) *For* $a, b \in S$*, we have* $\ell_{a\cdot\xi\cdot b} = \ell_a \circ \ell_\xi \circ \ell_b$*.*

Proof. Note that (i) gives a more direct proof of Theorem 6, while (ii) implies that the assignment $\xi \mapsto \ell_x$ is an embedding of (S, S)-bisets $\Omega \to {}_S\widehat{G}_S$.

(i) As F_1 is nondegenrate, each of the maps $\omega \mapsto F_1(1_\Omega, \omega)$ and $\omega \mapsto F_1(\xi, \omega)$ are bijections, so at the very least ℓ_ξ is a permutation of Ω. We must first show that it respects the right S-structure on Ω. As F_1 is the first coordinate of a Frobenius isomorphism $\Omega \overset{*}{\times} \Omega \cong \Omega \overset{\odot}{\times} \Omega$, we have

$$F_1(s_1 \cdot \omega_1 \cdot t, s_2 \cdot \omega_2 \cdot t) = s_1 \cdot F_1(\omega_1, \omega_2) \cdot s_1^{-1}$$

for all $(s_1, s_2) \in S \times S$, $t \in S$, and $\omega_1\omega_2 \in \Omega \times \Omega$. Thus, for $\omega \in \Omega$ and $b \in S$, we have

$$\begin{aligned} \ell_\xi(\omega \cdot b) &= F_1(\xi, F_1(1_\Omega, \omega \cdot b)) = F_1(\xi, F_1(1_\Omega \cdot b^{-1}, \omega)) \\ &= F_1(\xi, F_1(b^{-1} \cdot 1_\Omega, \omega)) = F_1(\xi, b^{-1} \cdot F_1(1_\Omega, \omega)) \\ &= F_1(\xi, F_1(1_\Omega, \omega)) \cdot b = \ell_\xi(\omega) \cdot b, \end{aligned}$$

so ℓ_ξ respects the right S-action. To see that ℓ_ξ give an isomorphism ${}_{S_\xi}\Omega_S \to {}^{c_\xi}_{S_\xi}\Omega_S$, we must show that for all $a \in S_\xi$ and $\omega \in \Omega$, we have

$$\ell_\xi(a \cdot \omega) = c_\xi(a) \cdot \ell_\xi(\omega).$$

Consider:

$$\begin{aligned} \ell_\xi(a \cdot \omega) &= F_1(\xi, F_1(1_\Omega, a \cdot \omega)) = F_1(\xi, F_1(1_\Omega, \omega) \cdot a^{-1}) \\ &= F_1(\xi \cdot a, F_1(1_\Omega, \omega)) = F_1(c_\xi(a) \cdot \xi, F_1(1_\Omega, \omega)) \\ &= c_\xi(a) \cdot F_1(\xi, F_1(1_\Omega, \omega)) = c_\xi(a) \cdot \ell_\xi(\omega), \end{aligned}$$

and we've proved the first result.

(ii) Now let $a, b \in S$ be arbitrary. That $\ell_{a\cdot\xi\cdot b} = \ell_a \circ \ell_\xi \circ \ell_b$ is equivalent to saying that for all $\omega \in \Omega$,

$$F_1(a \cdot \xi \cdot b, F_1(1_\Omega, \omega)) \overset{?}{=} a \cdot (\xi, F_1(1_\Omega, b \cdot \omega)),$$

but both of these are easily seen to be equal to

$$a \cdot F_1(\xi, F_1(1_\Omega, \omega) \cdot b^{-1})$$

by applying the equivariance property of F_1.

□

One might also ask whether, given a a nondegenerate Frobenius pairing F_1, we additionally have for all $\xi, \eta \in \Omega$,

$$\ell_\xi \circ \ell_\eta \overset{?}{=} \ell_{\ell_\xi(\eta)}.$$

This does not appear to be true in general, and seems to be closely related to the question of whether the multiplication on Ω defined by $(\xi, \omega) \mapsto \ell_\xi(\omega)$ is associative. Perhaps a better question would be whether, given a nondegenerate Frobenius pairing F_1, there exists *some* nondegenerate Frobenius pairing F_1^* that makes the multiplication associative. Of course, we know that this cannot exist in general, as that would imply that we have a group realizing $\mathcal{F}$. Most likely this means that we should relax our search: We do not seek nondegnerate Frobenius pairings, and should only be looking for "associativity" on the centric objects of Ω.

Let's therefore consider an arbitrary Frobenius pairing F_1 on Ω, and consider the effect of the pairing on point-stabilizers.

Proposition 12. *Suppose that $\omega_1, \omega_2 \in \Omega$ have point-stabilizers $(S_{\omega_1}, c_{\omega_1})$ and $(S_{\omega_2}, c_{\omega_2})$. Set $P = S_{\omega_1} \cap S_{\omega_2}$. Then the point-stabilizer of $F_1(\omega_1, \omega_2)$ contains $(c_{\omega_1} \circ c_{\omega_2}^{-1}, c_{\omega_2}P)$.*

Proof. We must show that for all $a \in P$,

$$c_{\omega_1}(a) \cdot F_1(\omega_1, \omega_2) = F_1(\omega_1, \omega_2) \cdot c_{\omega_2}(a).$$

Using the equivariance properties of F_1, we have

$$\begin{aligned} c_{\omega_1}(a) \cdot F_1(\omega_1, \omega_2) &= F_1(c_{\omega_1}(a) \cdot \omega_1, \omega_2) = F_1(\omega_1 \cdot a, \omega_2) \\ &= F_1(\omega_1, \omega_2 \cdot a^{-1}) = F_1(\omega_1, c_{\omega_2}(a^{-1}) \cdot \omega_2) = F_1(\omega_1, \omega_2) \cdot c_{\omega_2}(a), \end{aligned}$$

as desired. □

The importance of this claim comes from considering the minimal characteristic biset of a fusion system $\mathcal{F}$. We have already shown that the centric part of the MCB, $\Omega^c_{\mathcal{F}}$, can be described in terms of the nonextendable isomorphisms of $\mathcal{F}$ between centric subgroups.

Explicitly, let $\mathfrak{I}^c$ denote the set of nonextendable $\mathcal{F}$-isomorphisms between $\mathcal{F}$-centric subgroups. $\mathfrak{I}^c$ is naturally an (S, S)-biset by means of pre- and postcomposition with S-conjugation. Set $\overline{\mathfrak{I}}^c := S \backslash \mathfrak{I}^c / S$ to be the (S, S)-orbits of $\mathfrak{I}^c$. Note that for $\varphi, \psi \in \mathfrak{I}^c$ with sources P and Q, respectively, we have $[\varphi] = [\psi] \in \overline{\mathfrak{I}}^c$ if and only if the (S, S)-bisets $[\varphi, P]$ and $[\psi, Q]$ are isomorphic. We then have

$$\Omega^c_{\mathcal{F}} = \coprod_{[\varphi] \in \overline{\mathfrak{I}}^c} [\varphi, s(\varphi)],$$

where $s(\varphi)$ denotes the source of the isomorphism. This can be identified with the nonextendable isomorphisms of the linking system of $\mathcal{F}$, where the data of the composition has been forgotten.

We have the following fundamental property, due to Puig, of $\mathcal{F}$-morphisms between $\mathcal{F}$-centric subgroups:

Proposition 13. *Let $P \leq Q$ be $\mathcal{F}$-centric subgroups and $\psi_1, \psi_2 \in \mathcal{F}(Q, S)$. If $\psi_1|_P = \psi_2|_P$, then there exists $z \in Z(P)$ such that $\psi_2 = \psi_1 \circ c_z$.*

One can use this result to prove that if $\varphi \in \mathcal{F}(P, S)$ is an arbitrary morphism with $\mathcal{F}$-centric source, then there exists a unique nonextendable extension of φ, up to $(Z(\varphi P), Z(P))$-conjugacy. In particular, there is a unique element of $\overline{\mathfrak{I}}^c$ which contains all the nonextendable extensions of φ.

Suppose now that we have $\varphi : P \to Q$ and $\psi : P' \to Q'$ in $\mathfrak{I}^c$. If $P \cap P'$ is $\mathcal{F}$-centric, there is therefore a unique class in $\overline{\mathfrak{I}}^c$ that contains the nonextendable extensions of $\varphi \circ \psi^{-1} : \psi(P \cap P') \to \varphi(P \cap P')$. Our computation of point-stabilizers shows that if ω_φ induces φ and ω_ψ induces ψ, then for F_1 a Frobenius pairing on $\Omega_{\mathcal{F}}$ we have $F_1(\omega_\varphi, \omega_\psi)$ is contained in this class of $\overline{\mathfrak{I}}^c$. Thus a Frobenius pairing induces a partial multiplication on $\Omega^c_{\mathcal{F}}$ that behaves as we would expect on (S,S)-orbits. The only problem is that the multiplication need not be associative, which has always been the problem with constructing linking systems.

An alternate view of Frobenius reciprocity

Let's develop an alternate approach to Frobenius reciprocity, which will allow us to talk about higher dimensional analogues. For this section, we will use both perspectives of Ω as an (S,S)-biset or an $S \times S$-set, as well as higher analogues, interchangeably.

If Ω starts out life as an (S,S)-biset, it becomes an $S \times S$-set via

$$(s_1, s_2) \cdot \omega = s_1 \cdot \omega \cdot s_2^{-1}.$$

There is a natural extension of this action to $\Omega \times \Omega$ as an $S \times S \times S$-set, namely

$$(s_1, s_2, s_3) \odot (\omega_1, \omega_2) = (s_1 \cdot \omega_1 \cdot s_2^{-1}, s_2 \cdot \omega_2 \cdot s_3^{-1}).$$

More generally, Ω^n has the *natural* S^{n+1}-set action where the first copy of S act on the left of the first copy of Ω, the last copy of S acts on the right on the last copy of Ω, and the intermediate ith copy of S, $1 < i < n+1$, acts diagonally on the right on the $(i-1)$st copy of Ω and on the left on the ith copy:

$$(s_1, \dots, s_{n+1}) \odot (\omega_1, \dots, \omega_n) = (s_1 \cdot \omega_1 \cdot s_2^{-1}, s_2 \cdot \omega_2 \cdot s_3^{-1}, \dots, s_i \cdot \omega_i \cdot s_{i+1}^{-1}, \dots, s_n \cdot \omega_n \cdot s_{n+1}^{-1}).$$

We will keep the notation $\odot$ for this natural action.

Given the natural S^{n+1}-action on Ω^n, we can construct a new S^{n+1}-action given a permutation $\sigma \in \Sigma_{n+1}$ by first permuting the factors of S^{n+1} and then applying the natural action. Denote this new action by $\odot_\sigma$. Thus

$$(s_1, s_2, \dots, s_{n+1}) \odot_\sigma (\omega_1, \dots, \omega_n) := (s_{\sigma(1)}, s_{\sigma(2)}, \dots, s_{\sigma(n+1)}) \odot (\omega_1, \dots, \omega_n).$$

We can ask whether this new action is isomorphic to the natural one. We claim that Frobenius reciprocity is equivalent to $(\Omega^n, \odot) \cong (\Omega^n, \odot_\sigma)$ as S^{n+1}-sets for all $n \geq 1$ and $\sigma \in \Sigma_{n+1}$.

Consider first the case $n = 1$. Then the unique nonidentity element $(12) \in \Sigma_2$ twists the natural action by

$$(s_1, s_2) \odot_{(12)} \omega = (s_2, s_1) \odot \omega = s_2 \cdot \omega s_1^{-1},$$

which is clearly just the natural $S \times S$-action on Ω^{op}. Thus requiring $(\Omega, \odot) \cong (\Omega, \odot_{(12)})$ as $S \times S$-sets is exactly the same as saying that Ω is a symmetric (S,S)-biset.

Now consider the case $n = 2$ and the two actions of S^3 on Ω^2 we used to define Frobenius reciprocity. In terms of left S^3-actions, we have

$$(s_1, s_2, s_3) \odot (\omega_1, \omega_2) = (s_1 \cdot \omega_1 \cdot s_2^{-1}, s_2 \cdot \omega_2 \cdot s_3^{-1}) \text{ and}$$
$$(s_1, s_2, s_3) * (\omega_1, \omega_2) = (s_1 \cdot \omega_1 \cdot s_3^{-1}, s_2 \cdot \omega_2 \cdot s_3^{-1})$$

It is clear that the action $\odot$ used here is the same as the natural action of S^3 on Ω^2. The action $*$ is not quite $\odot_{(23)}$, but it related: It is $\odot_{(23)}$ together with the swap of S-actions on the second coordinate. If Ω is symmetric (as covered in the previous case), we may fix an isomorphism $\tau : \Omega \cong \Omega^{\mathrm{op}}$. Supposing that $(\Omega^2, \odot) \cong (\Omega^2, \odot_{(23)})$, the composite

$$(\Omega^2, \odot) \xrightarrow{\cong} (\Omega^2, \odot_{(23)}) \xrightarrow{\mathrm{id}_\Omega \times \tau} (\Omega^2, *)$$

yields an isomorphism implying that Ω is a Frobenius reciprocity biset.

Conversely, if Ω is symmetric and satisfies Frobenius reciprocity, $(\Omega^2, \odot) \cong (\Omega^2, \odot_\sigma)$ for all $\sigma \in \Sigma_3$. I further believe that $(\Omega^n, \odot) \cong (\Omega^n, \odot_\sigma)$ for all $\sigma \in \Sigma_{n+1}$. Consider:

Example 14. If Ω is a symmetric Frobenius reciprocity biset, then $(\Omega^3, \odot) \cong (\Omega^3, \odot_{(243)})$. Setting $\overline{\odot} := \odot_{(243)}$ and viewing $(\Omega^3, \overline{\odot})$ as an (S^3, S)-biset, we have

$$(s_1, s_2, s_3)\overline{\odot}(\omega_1, \omega_2, \omega_3)\overline{\odot}t = (s_1 \cdot \omega_1 \cdot t, t^{-1} \cdot \omega_2 \cdot s_2^{-1}, s_2 \cdot \omega_3 \cdot s_3^{-1}).$$

Then for any subgroup $(\varphi \times \psi \times \chi, P)$ with $P \le S$ and $\varphi, \psi, \chi \in \mathrm{Hom}(P, S)$, we have

$$\begin{aligned}
(\Omega^3, \overline{\odot})^{(\varphi\times\psi\times\chi, A)} &= \left\{ (\omega_1, \omega_2, \omega_3) \,\middle|\, \begin{array}{l} (\varphi(a), \psi(a), \chi(a))\overline{\odot}(\omega_1, \omega_2, \omega_3) = (\omega_1, \omega_2, \omega_3)\overline{\odot}a \\ \quad \forall\, a \in P \end{array} \right\} \\
&= \left\{ (\omega_1, \omega_2, \omega_3) \,\middle|\, \begin{array}{ll} \varphi(a) \cdot \omega_1 = \omega_1 \cdot a & \\ \omega_2 \cdot \psi(a)^{-1} = a^{-1} \cdot \omega_2 & \quad \forall\, a \in P \\ \psi(a) \cdot \omega_3 \cdot \chi(a)^{-1} = \omega_3 & \end{array} \right\} \\
&= \Omega^{(\varphi, P)} \times \Omega^{(\psi^{-1}, \psi P)} \times \Omega^{(\psi\chi^{-1}, \chi P)}
\end{aligned}$$

As Ω is a symmetric Frobenius reciprocity biset, it is $\mathcal{F}$-stable and $\mathcal{F}$-generated for $\mathcal{F} := \mathcal{F}_\Omega = \mathcal{F}^\Omega$. Thus the fixed-point set is empty unless each of φ, ψ, and χ are in $\mathcal{F}$, in which case these fixed-point factors have order $\left|\Omega^{(\iota, P)}\right|$. A similar computation yields

$$\left|(\Omega^3, \overline{\odot})^{(\varphi\times\psi\times\chi, P)}\right| = \left|(\Omega^3, \odot)^{(\varphi\times\psi\times\chi, P)}\right|$$

for all $\varphi, \psi, \chi \in \mathrm{Hom}(P, S)$, so we have our desired isomorphism of (S^3, S)-bisets.

Note that if $\Omega = G$ is a group, then this isomorphism can be realized by the map

$$(g, h, k) \mapsto (ghk, (hk)^{-1}, h),$$

so we can encode a version of the threefold product of elements of Ω by considering higher Frobenius reciprocity relations.

In fact, as the original example of the action $*$ on Ω^2 shows, we actually have even more than isomorphisms of actions given by permutations. We can generalize even further by considering the action $*$ on Ω^3, given by

$$(s_1, s_2, s_3) * (\omega_1, \omega_2, \omega_3) * t = (s_1 \cdot \omega_1 \cdot t, s_2 \cdot \omega_2 \cdot t, s_3 \cdot \omega_3 \cdot t),$$

and the same fixed-point computations will show that $(\Omega^3, *) \cong (\Omega^3, \odot)$, but $*$ cannot be realized by a permutation applied to $\odot$ (there are 3 occurrences of the action by t, while in $\odot$ each element acts at most twice). Per Sune's suggestion, it seems that one might prove a general theorem of the following form:

Theorem 15. *Let Ω be a symmetric Frobenius reciprocity biset. Let S^{n+1} act on Ω^n via $*$ in such a way that on a given factor Ω_i, the S^{n+1}-action is determined by precisely two factors of S. Let Γ be the graph with vertices the numbers $\{1, 2, \dots, n+1\}$ and an edge from i to j if there is a factor of Ω on which the two copies of S that act are the ith and jth factors. If Γ is a tree, then $(\Omega^n, *) \cong (\Omega^n, \odot)$.*

If we find an application for this result, I'll write out the statement and proof in more detail.