Fusion systems seminar 9:
Characteristic Bisets II

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Goals:

(i) Describe how a biset can give rise to a fusion system.
(ii) Show that the existence of a characteristic biset implies the saturation.
(iii) Use characteristic bisets to prove the saturation of K-normalizer subsystems.

We continue our exploration of bisets in fusion theory. Last time we introduced the notion of $\mathcal{F}$-characteristic biset and showed that every saturated fusion system possesses one. Our main result today is to show the reverse implication, that the existence of an $\mathcal{F}$-characteristic biset implies the saturation of $\mathcal{F}$. As a consequence we will give a cleaner proof of the saturation of K-normalizer subsystems.

The two fusion systems of a biset

We begin with a finite p-group S, and for the moment we will not consider a particular fusion system on S. Instead we will consider an (S,S)-biset Ω and show that, under suitable conditions, Ω gives rise to two related fusion systems on S. In the next section we will see that if we do have a particular fusion system $\mathcal{F}$ on S for which Ω is $\mathcal{F}$-characteristic, then the two fusion systems defined by Ω are equal and are both in fact $\mathcal{F}$ itself.

We start by assuming that Ω is bifree, so that every point-stabilizer is a twisted diagonal subgroup of $S\times S$. Recall our notational convention for $\omega\in\Omega$:

$$\begin{aligned}\mathrm{Stab}(\omega) &= \{(a,b)\in S\times S | a\cdot\omega=\omega\cdot b\}\\ &= (c_\omega, S_\omega) := \{(c_\omega(a),a) | a\in S_\omega\}.\end{aligned}$$

We think of the homomorphism $c_\omega : S_\omega\to S$ induced by ω as a "conjugation" morphism.

We also adopt the convention that $[\varphi,P]$ will mean an (S,S)-orbit containing a point with stabilizer (φ,P).

Definition 1. For $P,Q\leq S$, we write

$$\begin{aligned}\mathrm{Hom}_\Omega(P,Q) &= \{\varphi\in\mathrm{Hom}(P,Q) |\ \exists\ \omega\in\Omega \text{ s.th. } c_\omega|_P=\varphi\}\\ &= \{\varphi\in\mathrm{Hom}(P,Q) |\ \Omega^{(\varphi,P)}\neq\emptyset\}\end{aligned}$$

for the set of group maps induced by conjugation by elements of ω.

We would like to piece the various $\mathrm{Hom}_\Omega(P,Q)$ together as P and Q range over the subgroups of S to obtain a category, and hopefully even a fusion system. There are a couple of problems with this for general Ω, however. First, there is no reason to assume that id_P is induced by any element of Ω, and indeed it may not be. However, if $[\mathrm{id}_S,S]\subset\Omega$, then $\mathrm{id}_P\in\mathrm{Hom}_\Omega(P,P)$ for all $P\leq S$.

The second problem is that composition of group maps need not induce a map

$$\mathrm{Hom}_\Omega(Q,R)\times\mathrm{Hom}_\Omega(P,Q)\to\mathrm{Hom}_\Omega(P,R)$$

as required of a category. Instead of spelling out explicitly what condition we must impose on Ω at this point to make composition well-defined, we will simply accept that this is a potential failing that will have to be corrected later.

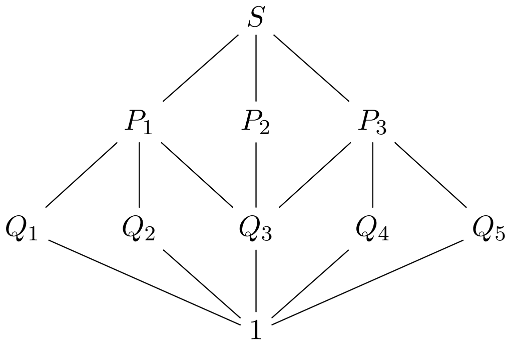
Definition 2. Given a bifree (S,S)-biset Ω containing $[\mathrm{id}_S, S]$, let $\mathcal{F}_\Omega$ denote the category-without-composites whose objects are the subgroups of S and whose hom-sets are given by

$$\mathcal{F}_\Omega(P,Q) = \mathrm{Hom}_\Omega(P,Q).$$

Even if $\mathcal{F}_\Omega$ is an honest category, it need not be a fusion system. The problem is that if $\varphi \in \mathrm{Hom}_\Omega(P,Q)$ is an isomorphism of groups, φ^{-1} need not be in $\mathrm{Hom}_\Omega(Q,P)$; this can occur if $[\varphi, P] \subseteq \Omega$ but $[\varphi^{-1}, Q] \nsubseteq \Omega$. Since $[\varphi, P]^{\mathrm{op}} = [\varphi^{-1}, \varphi P]$, this problem can be corrected by requiring $\Omega \cong \Omega^{\mathrm{op}}$, i.e., that Ω be symmetric.

Finally, even in the case when $\mathcal{F}_\Omega$ is a category and a fusion system, it need not be saturated. To see when we get saturation, we will need our second fusion system defined by Ω. First, an example to illustrate the difficulties:

Example 3. Consider (as always) $S = D_8$ and its subgroup lattice:



Here $P_1 \cong P_4 \cong C_2^2$ and all other proper subgroups are cyclic. For $i = 1, 3$, let $\alpha_i \in \mathrm{Aut}(P_i)$ be an automorphism of order 3, and for $1 \le i, j \le 5$ let $\beta_{ij} : Q_i \to Q_j$ be the unique isomorphism. We consider several (S,S)-bisets and the corresponding $\mathcal{F}_\Omega$:

$\Omega_0 =$ $[\mathrm{id}_1, 1]$. Thus Ω_0 is the free (not just bifree) (S,S)-biset, and none of its elements define conjugation on any nontrivial subgroup. Thus $\mathrm{Hom}_{\Omega_0}(P,Q) = \emptyset$ unless $P = Q = 1$, in which case it is the nonempty but uninteresting.

$\Omega_1 =$ $[\mathrm{id}_S, S]$. Thus Ω_1 is S as an (S,S)-biset. It is immediate that $\mathcal{F}_{\Omega_1} = \mathcal{F}_S(S)$ is the minimal saturated fusion system on S.

$\Omega_2 =$ $[\mathrm{id}_S, S] + [\alpha_1, P_1] + [\alpha_3, P_3]$. Note that if $\mathrm{Stab}(\omega) = (\alpha_1, P_1)$ and $a, b \in S$, then (since $P_1 \trianglelefteq S$) $\mathrm{Stab}(a \cdot \omega \cdot b) = (c_a \circ \alpha_1 \circ c_b, P_1)$. As a and b range over S, we realize every automorphism of P_1 in this manner; thus $\mathrm{Aut}_{\Omega_2}(P_1) = \mathrm{Aut}(P_1) \cong \Sigma_3$; similarly for P_3. However, $\mathcal{F}_{\Omega_2}$ is not a category, as it contains β_{13} and β_{35} but not $\beta_{15} = \beta_{35} \circ \beta_{13}$.

$\Omega_3 =$ $[\mathrm{id}_S, S] + [\beta_{13}, Q_1]$. Then $\mathrm{Hom}_{\Omega_2}(P,Q) = \mathrm{Hom}_S(P,Q)$ unless $P = Q_1$ or $P = Q_2$ and $Q \ge Q_3$, in which case β_{13} or β_{23} is also included. As $Q_3 = Z(S)$, $\varphi \circ \beta_{13} = \beta_{13}$ for all $\varphi \in \mathcal{F}_S(S)$, so in particular $\mathcal{F}_{\Omega_3}$ is a category. It is not, however, a fusion system, as it does not contain $\beta_{13}^{-1} = \beta_{31}$.

$\Omega_4 =$ $[\mathrm{id}_S, S] + [\beta_{13}, Q_1] + [\beta_{31}, Q_3]$. Now $\mathcal{F}_{\Omega_4}$ is a category and a fusion system, but it is not saturated: β_{13} is an isomorphism with fully centralized target, but β_{13} is not the restriction of any morphism with source P_1. Thus the extension axiom fails.

$\Omega_5 =$ $[\mathrm{id}_S, S] + [\alpha, S]$, where $\alpha \in \mathrm{Aut}(S)$ is of order 2 (it reflects the symmetry of the subgroup lattice). Then $\mathcal{F}_{\Omega_5}$ is again a nonsaturated fusion system: This time the extension axiom is realized, but $\mathrm{Inn}(S) \notin \mathrm{Syl}_2(\mathrm{Aut}_{\mathcal{F}_{\Omega_5}}(S))$, so the Sylow axiom fails.

$\Omega_6 =$ $[\mathrm{id}_S, S] + [\alpha_1, P_1]$. This time we have saturation, and in fact $\mathcal{F}_{\Omega_6} = \mathcal{F}_{D_8}(\Sigma_4)$. Moreover, $\Omega_6 \cong {}_S(\Sigma_4)_S$ is the minimal characteristic biset for $\mathcal{F}_{D_8}(\Sigma_4)$.

$\Omega_7 =$ $[\mathrm{id}_S, S]+[\alpha_1, P_1]+[\alpha_3, P_3]+[\beta_{15}, Q_1]+[\beta_{51}, Q_5]$. Again we have a saturated fusion system $\mathcal{F}_{\Omega_7} = \mathcal{F}_{D_8}(A_6)$, and this is the minimal characteristic biset for its fusion system. Note however that this is *not* A_6 as a (D_8, D_8)-biset, and in fact does not come from any group. It is, however, *contained* in A_6, or indeed any group that induces this fusion system on D_8.

We now turn to the second fusion system derived from Ω.

Definition 4. Let $\widehat{G} = \widehat{G}_\Omega$ be the group $\mathrm{Aut}({}_1\Omega_S)$ of permutations of Ω that respect the right S-set structure. For $a \in S$, denote by ℓ_a the permutation of Ω induced by left multiplication by a. As the left and right actions of S on Ω commute, $\ell_a \in \widehat{G}$, and as the left S-action is free we have $\ell : S \to \widehat{G}$ is an embedding of S as a (non-Sylow) p-subgroup. Write $\widehat{S}$ for the image of S in $\widehat{G}$.

Set $\mathcal{F}^\Omega = \mathcal{F}_{\widehat{S}}(\widehat{G})$. We view $\mathcal{F}^\Omega$ as a fusion system on S via the identification $\ell : S \cong \widehat{S}$.

Thus $\mathcal{F}^\Omega$ is always a fusion system on S, though as $\widehat{S}$ is not in general Sylow in $\widehat{\Omega}$, it need not be saturated. We can describe $\widehat{G}$ more explicitly by observing that any element $\sigma \in \widehat{G}$ induces and is determined by a permutation on the right S-orbits Ω/S together with a right S-map $S_S \to S_S$ for each S-orbit, and that all possible collections of permutation and S-maps can be realized. As $\mathrm{Aut}_S(S_S) \cong S$, we have

$$\widehat{G} \cong S \wr \Sigma_{\Omega/S} = (S)^{|\Omega/S|} \rtimes \Sigma_{\Omega/S},$$

so $|\widehat{G}| = |S|^{|\Omega/S|} \cdot |\Omega/S|!$, and the p-part of the index $[\widehat{G} : \widehat{S}]$ is generally enormous. For example, the (D_8, D_8)-biset Ω_7 above, the minimal characteristic biset for $\mathcal{F}_{D_8}(A_6)$ has corresponding group $\widehat{G}_{\Omega_7}$ of order $8^{13} \cdot 13!$, so a Sylow 2-subgroup has order 2^{49}. Nevertheless, we will see that for certain well-chosen Ω, the fusion system $\mathcal{F}^\Omega$ will be saturated. This will require relating $\mathcal{F}^\Omega$ to $\mathcal{F}_\Omega$, but first we should give a more tractable description of the morphisms of $\mathcal{F}^\Omega$.

Lemma 5. *For $\varphi \in \mathrm{Hom}(P, Q)$, we have $\varphi \in \mathcal{F}^\Omega(P, Q)$ iff ${}_P\Omega_S \cong {}^{\varphi}_{P}\Omega_S$ as (P, S)-bisets.*

Proof. $\varphi \in \mathcal{F}^\Omega(P, Q)$ iff there is some $\sigma \in \widehat{G}$ such that $\varphi = c_\sigma|_{\widehat{P}}$, or $\sigma \circ \ell_a \circ \sigma^{-1} = \ell_{\varphi(a)}$ for all $a \in P$. We can rewrite this as

$$\sigma(a \cdot \omega) = \varphi(a) \cdot \sigma(\omega)$$

for all $a \in P$, $\omega \in \Omega$. Equivalently, σ is our desired isomorphism ${}_P\Omega_S \cong {}^{\varphi}_{P}\Omega_S$. □

Here is the first connection between $\mathcal{F}_\Omega$ and $\mathcal{F}^\Omega$:

Proposition 6. *For $P, Q, R \leq S$, consider the group maps*

$$P \xrightarrow{\varphi} Q \xrightarrow{\psi} R.$$

If $\varphi \in \mathcal{F}_\Omega(P, Q)$ and $\psi \in \mathcal{F}^\Omega(Q, R)$, then $\psi\varphi \in \mathcal{F}_\Omega(P, R)$. In other words, composition induces a pairing

$$\mathcal{F}^\Omega(Q, R) \times \mathcal{F}_\Omega(P, Q) \to \mathcal{F}_\Omega(P, R).$$

Proof. If $\varphi \in \mathcal{F}_\Omega(P, Q)$, there is some $\omega_0 \in \Omega$ such that $c_{\omega_0}|_P = \varphi$. Equivalently,

$$\varphi(a) \cdot \omega_0 = \omega_0 \cdot a$$

for all $a \in P$. By the previous lemma, if $\psi \in \mathcal{F}^{\Omega}(Q,R)$, there is some $\sigma \in \widehat{G}$ such that

$$\sigma(b \cdot \omega \cdot c) = \psi(b) \cdot \sigma(\omega) \cdot c$$

for all $b \in Q$, $c \in S$, and $\omega \in \Omega$. We then have for all $a \in P$

$$\psi\varphi(a) \cdot \sigma(\omega_0) = \sigma(\varphi(a) \cdot \omega_0) = \sigma(\omega_0 \cdot a) = \sigma(\omega_0) \cdot a,$$

where the first equality follows from the assumption that $\varphi(a) \in Q$. In other words, $c_{\sigma(\omega_0)}|_P = \psi\varphi$, so that $\psi\varphi \in \mathcal{F}_{\Omega}(P,R)$. □

Corollary 7. *If* $[\mathrm{id}_S, S] \subseteq \Omega$*, then* $\mathcal{F}^{\Omega} \subseteq \mathcal{F}_{\Omega}$*.*

Proof. If $[\mathrm{id}_S, S] \subseteq \Omega$, we have $\mathrm{id}_P \in \mathcal{F}_{\Omega}(P,P)$ for all $P \leq S$. For any $\varphi \in \mathcal{F}^{\Omega}(P,Q)$, we have $\varphi = \varphi \circ \mathrm{id}_P$ expresses φ as the composite $\mathcal{F}^{\Omega}(P,Q) \times \mathcal{F}_{\Omega}(P,P) \to \mathcal{F}_{\Omega}(P,Q)$. The result follows from the previous Proposition. □

Our next goal then becomes to understand when the reverse containment is true.

Bisets and saturation

Now let $\mathcal{F}$ be an abstract fusion system on S and Ω a bifree symmetric (S,S)-biset. Recall that Ω is *$\mathcal{F}$-generated* if $c_\omega \in \mathcal{F}(S_\omega, S)$ for all $\omega \in \Omega$, and Ω is *$\mathcal{F}$-stable* if for all $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$ we have $|\Omega^{(\varphi,P)}| = |\Omega^{(\iota,P)}|$, or equivalently ${}_P\Omega_S \cong {}^{\varphi}_{P}\Omega_S$ as (P,S)-bisets (using the fact that the symmetry of Ω implies the corresponding right-sided versions). We then have as a direct consequence of the definitions and Corollary 7:

Proposition 8.

- *If* Ω *is* $\mathcal{F}$*-generated, then* $\mathcal{F}_{\Omega} \subseteq \mathcal{F}$*.*
- *If* Ω *is* $\mathcal{F}$*-stable, then* $\mathcal{F} \subseteq \mathcal{F}^{\Omega}$*.*
- *If* Ω *is* $\mathcal{F}$*-generated,* $\mathcal{F}$*-stable, and contains* $[\mathrm{id}_S, S]$*, then*

$$\mathcal{F}_{\Omega} = \mathcal{F} = \mathcal{F}^{\Omega}.$$

In particular, $\mathcal{F}_{\Omega}$ *is a fusion system.*

Recall that the bifree symmetric (S,S)-biset Ω is *$\mathcal{F}$-characteristic* if Ω is $\mathcal{F}$-generated, $\mathcal{F}$-stable, and $p \nmid |S\backslash\Omega|$. It is easy to see that $\mathcal{F}$-stability and the order condition imply that any $\mathcal{F}$-characteristic biset must contain a copy of $[\mathrm{id}_S, S]$. Thus Proposition 8 says that if Ω is *almost* $\mathcal{F}$-characteristic, in that it satisfies all the conditions except possibly $p \nmid |S\backslash\Omega|$, then $\mathcal{F}$ agrees with $\mathcal{F}_{\Omega}$ and $\mathcal{F}^{\Omega}$. This does not say that any of these fusion systems are saturated, but once we impose the final order condition, it will.

Theorem 9. *If there exists an* $\mathcal{F}$*-characteristic biset* Ω*, then* $\mathcal{F}$ *is saturated.*

Before we prove the saturation axioms, we set up our notation and establish some basic results. Suppose from now on that Ω is an $\mathcal{F}$-characteristic biset, so that $\mathcal{F}_{\Omega} = \mathcal{F} = \mathcal{F}^{\Omega}$ and $p \nmid |S\backslash\Omega|$. We will freely make use of the equality of the three fusion systems to establish various facts, and generally refer to all three as $\mathcal{F}$.

As the saturation axioms deal with fully $\mathcal{F}$-normalized and $\mathcal{F}$-centralized $\mathcal{F}$-conjugates of the subgroups $P \leq S$, it will be useful to consider the restricted (S,P)-biset ${}_S\Omega_P$. The (S,P)-point-stabilizer of $\omega \in {}_S\Omega_P$ is a twisted diagonal subgroup of the form

$$(\varphi, A) \leq S \times P,$$

where $A \leq P$ and $\varphi \in \mathcal{F}(A,S)$. We will concentrate on those points for which $A = P$.

Notation 10. Let Ω^P denote the subset of Ω consisting of those elements that conjugate P into S. Explicitly, $\Omega^P = \{\omega \in \Omega | P \leq S_\omega\}$. We can also express Ω^P in terms of fixed points of twisted diagonal subgroups:

$$\Omega^P = \coprod_{\varphi \in \mathcal{F}(P,S)} \Omega^{(\varphi,P)}.$$

Lemma 11. $S\backslash\Omega^P = (S\backslash\Omega)^P$. *Consequently,* $p \nmid |S\backslash\Omega^P|$.

Proof. In other words, we can also identify Ω^P as the set of elements of Ω whose image in $S\backslash\Omega$ are fixed by the right P-action. If $\omega \in \Omega^P$, then for all $a \in P$ we have $c_\omega(a)\cdot\omega = \omega\cdot a$, so $S\cdot\omega = S\cdot\omega\cdot a$ and $S\cdot\omega \in (S\backslash\Omega)^P$. The converse implication is similarly obvious.

Since $p \nmid |S\backslash\Omega|$ and P is a p-group, the second claim follows immediately. □

When we view Ω^P as an (S,P)-biset ${}_S\Omega^P_P$, every point-stabilizer will be of the form (φ, P) for $\varphi \in \mathcal{F}(P,S)$. The twisted diagonal subgroups (φ,P) and (ψ,P) determine isomorphic (S,P)-orbits iff $\psi = c_x \circ \varphi$ for some $x \in S$, or equivalently

$$[\varphi] = [\psi] \in \mathrm{Rep}_{\mathcal{F}}(P,S) := \mathrm{Inn}(S)\backslash\mathcal{F}(P,S).$$

Thus when we decompose ${}_S\Omega^P_P$ into (S,P)-orbits, there are integers $c([\varphi],P)$ such that

$${}_S\Omega^P_P = \coprod_{[\varphi]\in\mathrm{Rep}_{\mathcal{F}}(P,S)} c([\varphi],P)\cdot[\varphi,P],$$

where $[\varphi,P]$ is the (S,P)-orbit with (φ,P) occurring as a point-stabilizer.

Lemma 12. $c([\varphi],P) = \frac{\Omega^{(\iota,P)}}{|C_S(\varphi P)|}$. *Consequently,* φP *is fully* $\mathcal{F}$*-centralized iff* $p \nmid c([\varphi],P)$.

Proof. For all $x \in \Omega^P$, we have $\mathrm{Stab}(x\cdot\omega) = (c_x\circ\omega, P)$. Therefore

$$\left|[\psi,P]^{(\varphi,P)}\right| = \begin{cases} |C_S(\varphi P)| & \text{if } [\varphi]=[\psi]\in\mathrm{Rep}_{\mathcal{F}}(P,S), \\ 0 & \text{else.}\end{cases}$$

Noting that $(\Omega^P)^{(\varphi,P)} = \Omega^{(\varphi,P)}$ and using the $\mathcal{F}$-stability of Ω, we have

$$\left|\Omega^{(\iota,P)}\right| = \left|\Omega^{(\varphi,P)}\right| = c([\varphi],P)\cdot|C_S(\varphi P)|,$$

and the first claim follows.

Thus $c([\varphi],P)$ depends only on the order of the S-centralizer of φP, and

$$c([\psi],P) = \frac{|C_S(\varphi P)|}{|C_S(\psi P)|}\cdot c([\varphi],P)$$

for all $\varphi,\psi \in \mathcal{F}(P,S)$. In particular, $c([\psi],P) \equiv_p 0$ whenever ψP is not fully $\mathcal{F}$-centralized.

Since $|[\varphi,P]| = |S|$ and we know that $p \nmid |S\backslash\Omega^P|$, we therefore have

$$0 \not\equiv_p |S\backslash\Omega^P| = \sum_{[\varphi]\in\mathrm{Rep}_{\mathcal{F}}(P,S)} c([\varphi],P) \equiv_p \sum_{\substack{[\varphi]\in\mathrm{Rep}_{\mathcal{F}}(P,S)\\ \varphi P\in\mathcal{F}^{fc}}} c([\varphi],P)$$

where $\mathcal{F}^{fc}$ is the set of fully $\mathcal{F}$-centralized subgroups of S. Since $c([\varphi],P)$ is constant on those $[\varphi]$ with fully $\mathcal{F}$-centralized image, $p \nmid c([\varphi],P)$ iff φP is fully $\mathcal{F}$-centralized. □

Proof of Theorem 9. We must prove:

(i) If P is fully $\mathcal{F}$-normalized, $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$ and P is fully $\mathcal{F}$-centralized.
(ii) Every $\varphi \in \mathcal{F}(P,S)$ with φP fully $\mathcal{F}$-centralized has an extension $\widetilde{\varphi} \in \mathcal{F}(N_\varphi, S)$.

(i) Suppose that P is fully $\mathcal{F}$-normalized and let k_P denote the number of S-conjugacy classes of fully $\mathcal{F}$-normalized $\mathcal{F}$-conjugates of P. We compute

$$0 \not\equiv_p \left|S\backslash\Omega^P\right| = \frac{1}{|S|}\sum_{\varphi\in\mathcal{F}(P,S)}\left|\Omega^{(\varphi,P)}\right| = \frac{1}{|S|}\sum_{Q\cong_{\mathcal{F}}P}|\mathcal{F}(P,Q)|\cdot\left|\Omega^{(\iota,P)}\right|$$

$$= \frac{1}{|S|}\sum_{[Q]_S\cong_{\mathcal{F}}P}\frac{|S|}{|N_SQ|}\cdot|\operatorname{Aut}_{\mathcal{F}}(P)|\cdot\left|\Omega^{(\iota,P)}\right| = \frac{1}{|S|}\sum_{[Q]_S\cong_{\mathcal{F}}P}|S|\cdot\frac{|\operatorname{Aut}_{\mathcal{F}}(Q)|}{|N_SQ/C_SQ|}\cdot\frac{\left|\Omega^{(\iota,P)}\right|}{|C_SQ|}$$

$$= \sum_{[Q]_S\cong_{\mathcal{F}}P}[\operatorname{Aut}_{\mathcal{F}}(Q):\operatorname{Aut}_S(Q)]\cdot\frac{\left|\Omega^{(\iota,P)}\right|}{|C_SP|} \equiv_p \sum_{\substack{[Q]_S\cong_{\mathcal{F}}P\\ Q\in\mathcal{F}^{fn}}}[\operatorname{Aut}_{\mathcal{F}}(P):\operatorname{Aut}_S(P)]\cdot c([\iota],P)$$

$$= k_P\cdot[\operatorname{Aut}_{\mathcal{F}}(P):\operatorname{Aut}_S(P)]\cdot c([\iota],P).$$

Perhaps some explanation is in order. On the first line, the second equality is the $\mathcal{F}$-stability of Ω together with a partition of $\mathcal{F}(P,Q)$ by image of the morphism.

On the second line, $[Q]_S$ denotes the S-conjugacy class of $Q \leq S$, so the first sum is taken over the S-conjugacy classes of subgroups $\mathcal{F}$-conjugate to P. The term $\frac{|S|}{|N_SQ|}$ is the number of elements of $[Q]_S$, and we're replacing $|\mathcal{F}(P,Q)|$ by $|\operatorname{Aut}_{\mathcal{F}}(P)|$ because $Q \cong_{\mathcal{F}} P$. Note in particular that the summand depends only on $|N_SQ|$, which we use in the second equality of the third line (where it appears that the summand may also depend on $|C_SP|$).

On the third line, we note that after canceling the two copies of $|S|$, we are still left with a product of integers. Since the summands depend only on $|N_SQ|$ and each non fully $\mathcal{F}$-normalized subgroup is a power of p times the summand for P, we may ignore all but the fully $\mathcal{F}$-normalized subgroups modulo p.

On the final line we use Lemma 12 for the equality $\frac{|\Omega^{(\iota,P)}|}{|C_SP|} = c([\iota],P)$.

To recap, we have that if P is fully $\mathcal{F}$-normalized, then

$$0 \not\equiv_p k_P\cdot[\operatorname{Aut}_{\mathcal{F}}(P):\operatorname{Aut}_S(P)]\cdot c([\iota],P).$$

Therefore none of the three integers k_P, $[\operatorname{Aut}_{\mathcal{F}}(P):\operatorname{Aut}_S(P)]$, or $c([\iota],P)$ are divisible by p. That $p \nmid c([\iota],P)$ implies by Lemma 12 that P is fully $\mathcal{F}$-centralized, and that $p \nmid [\operatorname{Aut}_{\mathcal{F}}(P):\operatorname{Aut}_S(P)]$ means $\operatorname{Aut}_S(P) \in \operatorname{Syl}_p(\operatorname{Aut}_{\mathcal{F}}(P))$. This was our goal.

Aside: Note that we've actually proved a bit more than we set out to, in that we now know that k_P is prime to p. Since we've already shown that every saturated fusion system has a characteristic idempotent, we see that for saturated $\mathcal{F}$, the number of S-conjugacy classes of fully $\mathcal{F}$-normalized $\mathcal{F}$-conjugates to a given subgroup is always prime to p.

(ii) Suppose that $\varphi \in \mathcal{F}(P,Q)_{\mathrm{iso}}$ with Q fully $\mathcal{F}$-centralized. The set of all elements of Ω that induce φ is $\Omega^{(\varphi,P)}$; our goal is to find some $\omega_0 \in \Omega^{(\varphi,P)}$ such that $N_\varphi \leq S_{\omega_0}$. We consider the action of $N := N_{S\times S}((\varphi,P))$ on $\Omega^{(\varphi,P)}$ because of the following relationship between the extender N_φ and N:

For $(m,n) \in N$ and every $a \in P$, we have

$$c_{(m,n)}(\varphi(a),a) = (c_m(\varphi(a)),c_n(a)) = (\varphi(b),b)$$

for some $b \in P$. Thus for all $a \in P$ we have $c_m \circ \varphi(a) = \varphi \circ c_n(a)$, so that $\varphi \circ c_n \circ \varphi^{-1} \in \operatorname{Aut}_S(P)$. It follows that $N_\varphi = \operatorname{pr}_2(N)$, i.e., the extender is the projection onto the second coordinate of $N \leq S \times S$.

It is easy to see that $\ker(\mathrm{pr}_2) = C_S(\varphi P) \times 1$, so $C_S(\varphi P) \times 1 \trianglelefteq N$ and N acts on $C_S(\varphi P)\backslash\Omega^{(\varphi,P)}$. We have

$$|C_S(\varphi P)\backslash\Omega^{(\varphi,P)}| = \frac{|\Omega^{(\varphi,P)}|}{|C_S(\varphi P)|} = \frac{|\Omega^{(\iota,P)}|}{|C_S(\varphi P)|} = c([\varphi], P),$$

which is prime to p as φP is assumed fully $\mathcal{F}$-centralized. Therefore there is a fixed point $C_S(\varphi P) \cdot \omega_0$ of the N-action on $C_S(\varphi P)\backslash\Omega^{(\varphi,P)}$. Thus for all $(m, n) \in N$, there is some $z \in C_S(\varphi P)$ such that

$$m \cdot \omega_0 \cdot n^{-1} = z \cdot \omega_0, \qquad \text{or} \qquad z^{-1}m \cdot \omega_0 = \omega_0 \cdot n,$$

so that $(z^{-1}m, n) \in \mathrm{Stab}(\omega_0)$. Since we can find such an element of $C_S(\varphi P)$ for every $(m, n) \in N$, it follows that $N_\varphi = \mathrm{pr}_2(N) \leq S_{\omega_0}$. Thus c_{ω_0} is our desired extension of φ, and we are done.

□

Characteristic bisets and K-normalizers

We now use our characterization of saturation in terms of characteristic bisets to prove that K-normalizer subsystems are saturated. We recall the basic definitions:

Definition 13. Let $\mathcal{F}$ be a saturated fusion system on S, P subgroup of S, and K a subgroup of $\mathrm{Aut}(P)$.

- The *K-normalizer of P in S* is $N_S^K P := \{n \in N_S P \big| c_n|_P \in K\}$.
- $\mathrm{Aut}_S^K(P) = \mathrm{Aut}_S(P) \cap K \cong N_S^K P / C_S P$.
- $\mathrm{Aut}_\mathcal{F}^K(P) = \mathrm{Aut}_\mathcal{F}(P) \cap K \geq \mathrm{Aut}_S^K(P)$.
- For any $\varphi \in \mathcal{F}(P, S)$, set ${}^\varphi K = \{\varphi \circ \alpha \circ \varphi^{-1}\} \leq \mathrm{Aut}(\varphi P)$.
- P is *fully K-normalized in $\mathcal{F}$* if $|N_S^K P| \geq |N_S^{{}^\varphi K} \varphi P|$ for all $\varphi \in \mathcal{F}(P, S)$.
- The *K-normalizer fusion subsystem of P in $\mathcal{F}$* is the fusion system $N_\mathcal{F}^K P$ on $N_S^K P$ whose morphisms from A to B defined to be
$$\{\varphi \in \mathcal{F}(A, B) \big| \, \exists \, \widetilde{\varphi} \in \mathcal{F}(PA, PB) \text{ s.th. } \widetilde{\varphi}|_A = \varphi \text{ and } \widetilde{\varphi}|_P \in K\}.$$

Our main goal is to prove the following:

Theorem 14. *If P is fully K-normalized in $\mathcal{F}$, then $N_\mathcal{F}^K P$ saturated.*

If Ω is an $\mathcal{F}$-characteristic biset, we introduce the following:

Definition 15. The *K-normalizer subbiset of P in Ω* is

$$N_\Omega^K P := \{\omega \in \Omega \big| P \leq S_\omega \text{ and } c_\omega|_P \in K\}.$$

We also have the useful alternate description:

$$N_\Omega^K P = \coprod_{\alpha \in K} \Omega^{(\alpha,P)}.$$

Note that $N_\Omega^K P$ is *not* an (S, S)-biset. However, the computation

$$\mathrm{Stab}(a \cdot \omega \cdot b) = (c_a \circ c_\omega \circ c_b, S_\omega^b)$$

implies that it is naturally an $(N_S^K P, N_S^K P)$-biset.

Before diving in to the proof, we mention a few technical lemmas we will need.

Lemma 16. *If $\omega \in N_\Omega^K P$ has (S, S)-stabilizer (c_ω, S_ω), it has $(N_S^K P, N_S^K P)$-stabilizer*

$$(c_\omega|_{S_\omega \cap N_S^K P}, S_\omega \cap N_S^K P).$$

Proof. The only thing to show is that $c_\omega(S_\omega \cap N_S^K P) \leq N_S^K P$. Suppose $n \in S_\omega \cap N_S^K P$. Since c_ω is defined on P, it follows that $c_\omega(n) \leq N_S P$ as well, so we just need to show that $c_{c_\omega(n)}|_P \in K$. But $c_{c_\omega(n)} = c_\omega \circ c_n \circ c_\omega^{-1}$ on P. Since $c_n|_P \in K$, the result follows. □

Lemma 17. *P is fully K-normalized in $\mathcal{F}$ iff P is $\mathrm{Aut}_S^K(P) \in \mathrm{Syl}_p(\mathrm{Aut}_\mathcal{F}^K(P))$ and P is fully K-centralized in $\mathcal{F}$.*

Non-proof. I won't prove this here, because it's just a messy application of Sylow's Theorem. I suspect there is a biset-theoretic proof, analogous to the proof of Theorem 9, but I haven't been able to see it. However, we already have this result in the cases of interest: When $K = \mathrm{Aut}(P)$ we get $N_\mathcal{F} P$, and when $K = \{\mathrm{id}_P\}$ we get $C_\mathcal{F} P$. The first case is in the saturation axioms, the second is tautological. □

Proof of Theorem 14. We will show that $N_\Omega^K P$ is $N_\mathcal{F}^K P$-characteristic, which will imply the result by Theorem 9. We must show:

(i) $N_\Omega^K P$ is $N_\mathcal{F}^K P$-generated.
(ii) $N_\Omega^K P$ is $N_\mathcal{F}^K P$-stable.
(iii) $p \nmid |N_S^K P \backslash N_\Omega^K P|$.

(i) This follows immediately from Lemma 16.
(ii) We must show that for all $A \leq N_S^K P$ and $\varphi \in N_\mathcal{F}^K(A, N_S^K P)$,

$$\left|(N_\Omega^K P)^{(\varphi,A)}\right| = \left|(N_\Omega^K P)^{(\iota,A)}\right| = \left|(N_\Omega^K P)^{(\varphi^{-1},\varphi A)}\right|.$$

If $P \leq A$, then $(N_\Omega^K P)^{(\varphi,A)} = \Omega^{(\varphi,A)}$, and the equality follows from the $\mathcal{F}$-stability of Ω. For general A, let $\{\widetilde{\varphi}_i\}_{i=1}^n$ be the set of extensions of φ to a morphism in $N_\mathcal{F}^K P(PA, N_S^K P)$. I claim that

$$(N_\Omega^K P)^{(\varphi,A)} = \coprod_{i=1}^n (N_\Omega^K P)^{(\widetilde{\varphi}_i, PA)}.$$

Clearly the union is disjoint and we have the containment $\supseteq$. Conversely, given $\omega \in (N_\Omega^K P)^{(\varphi,A)}$, we consider the (S,S)-stabilizer (c_ω, S_ω) of ω. Since $\omega \in N_\Omega^K$, we have $P \leq S_\omega$, and since ω is fixed by (φ, A), we must have $A \leq S_\omega$ as well. Then $c_\omega|_{PA}$ is an extension of φ, so must be one of the $\widetilde{\varphi}_i$. The result follows from summing the orders of the $(\widetilde{\varphi}_i, PA)$-fixed points of $N_\Omega^K P$ as in the first case.

(iii) We compute

$$\begin{aligned} |N_S^K P \backslash N_\Omega^K P| &= \frac{1}{|N_S^K P|} \cdot \sum_{\alpha \in K} \left|\Omega^{(\alpha,P)}\right| = \frac{1}{|N_S^K P|} \cdot |\mathrm{Aut}_\mathcal{F}^K(P)| \cdot \left|\Omega^{(\iota,P)}\right| \\ &= \frac{|\mathrm{Aut}_\mathcal{F}^K(P)|}{|N_S^K P / C_S P|} \cdot \frac{\left|\Omega^{(\iota,P)}\right|}{|C_S P|} = [\mathrm{Aut}_\mathcal{F}^K(P) : \mathrm{Aut}_S^K(P)] \cdot c([\iota], P), \end{aligned}$$

where the first equality is our description of $N_\Omega^K P$ in terms of fixed points, the second follows from the $\mathcal{F}$-stability of Ω, and the last two are obvious. By Lemma 17, P fully K-normalized in $\mathcal{F}$ implies $\mathrm{Aut}_S^K(P) \in \mathrm{Syl}_p(\mathrm{Aut}_\mathcal{F}^K(P))$ and P is fully $\mathcal{F}$-centralized. Thus the index is prime to p and $c([\iota], P)$ is as well by Lemma 12.

□