Fusion systems seminar 8:
Characteristic Bisets I

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Date: 15/12/15

Goals:

(i) Introduce the notion of $\mathcal{F}$-characteristic biset.
(ii) Prove the freeness of the monoid of $\mathcal{F}$-sets.
(iii) Parameterize the $\mathcal{F}$-semicharacteristic biset when $\mathcal{F}$ is saturated.
(iv) Prove that $\mathcal{F}$ saturated implies the existence of an $\mathcal{F}$-characteristic biset.

Fusion systems as we have introduced them arise by considering the conjugation action of G on its p-subgroups (or Brauer pairs). In this talk we'll reverse that perspective: If $S \in \mathrm{Syl}_p(G)$, then S acts on G by left and right multiplication. The resulting structure is an (S,S)-*biset*, and it turns out to contain the data of $\mathcal{F}_S(G)$. We explore these ideas today.

Review of bisets and motivating example

Let S be a p-group and Ω an (S,S)-biset. For our motivating running example, let's assume we also have a finite group G with $S \in \mathrm{Syl}_p(G)$. We can view G as an (S,S)-biset, which we will denote ${}_SG_S$. We investigate the properties of ${}_SG_S$, and translate them to our more general (S,S)-biset Ω.

The most basic property of ${}_SG_S$ is that the left and right S-actions, considered individually, are both free. We thus impose our first condition:

Assumption 1. Ω is a bifree (S,S)-biset.

As $S \leq G$ is a subgroup, we obtain an inclusion ${}_SS_S \subseteq {}_SG_S$. Note that ${}_SS_S$ is transitive as an (S,S)-biset.

Assumption 2. Ω contains an (S,S)-orbit isomorphic to ${}_SS_S$.

The next part of the structure of ${}_SG_S$ we wish to emulate comes from the inversion anti-automorphism $?^{-1} : G \to G : g \mapsto g^{-1}$. Since $(agb)^{-1} = b^{-1}g^{-1}a^{-1}$ for all $g \in G$, $a,b \in S$, we see that inversion has the effect of swapping the left and right S-actions.

Definition 3. The *opposite biset* of Ω is the (S,S)-biset Ω^{op} whose underlying set is Ω and whose (S,S)-action $\odot$ is given by

$$a \odot \omega \odot b = b^{-1} \cdot \omega \cdot a^{-1}$$

for all $\omega \in \Omega$ and $a,b \in S$, where $\cdot$ is the original (S,S)-action on Ω.

Ω is *symmetric* if $\Omega \cong \Omega^{\mathrm{op}}$ as (S,S)-bisets.

Assumption 4. Ω is symmetric.

The remaining properties of ${}_SG_S$ are somewhat more subtle, but are what connect the biset ${}_SG_S$ with the fusion system $\mathcal{F}_S(G)$. These conditions are expressed in terms of point-stabilizers and fixed point sets.

Notation 5. For $\omega \in \Omega$, the (S,S)-*stabilizer* of ω is

$$\mathrm{Stab}_{(S,S)}(\omega) = \{(a,b) \in S \times S \,|\, a \cdot \omega = \omega \cdot b\}.$$

In the case that groups acting are clear from the context, we will simply write $\mathrm{Stab}(\omega)$.

Recall from our earlier discussion on the subgroups of product groups that the bifreeness of Ω has strong implications on the structure of point stabilizers.

Notation 6. For $P \leq S$ and $\varphi : P \hookrightarrow S$ an injection, the *left and right twisted diagonal subgroups of P along φ* are, respectively:

$$_{(P,\varphi)}\Delta = \{(\varphi(a), a) \big| a \in P\} \qquad \text{and} \qquad \Delta_{(P,\varphi)} = \{(a, \varphi(a)) \big| a \in P\}.$$

Note that $_{(P,\varphi)}\Delta = \Delta_{(\varphi P, \varphi^{-1})}$, so we could write everything in terms of either left or right twisted diagonal subgroups; we choose to work with the left variant. The motivating example of G explains this choice:

For $g \in G$, we have

$$\mathrm{Stab}_{(S,S)}(g) = \{(a,b) \in S \times S \big| ag = gb\} = \{(a,b) \in S \times S \big| a = gbg^{-1}\}.$$

Thus the first coordinate of an element of $\mathrm{Stab}_{(S,S)}(g)$ is c_g applied to the second, so representing the point-stabilizer as a left twisted diagonal subgroup is the natural choice. [Our initial decision to work with left-conjugation is really all that's at play here.] Moreover, a point $b \in S$ occurs in the second coordinate of $\mathrm{Stab}_{(S,S)}(g)$ iff $c_g(b) \in S$. Therefore the projection onto the second coordinate of the point-stabilizer yields a surjection onto $S_g := S \cap S^g$, the largest subgroup of S conjugated into S by g:

$$\mathrm{Stab}_{(S,S)}(g) = {}_{(S_g, c_g)}\Delta.$$

To simplify the notation, we introduce the following:

Notation 7. For $P \leq S$ and $\varphi : P \hookrightarrow S$, let (φ, P) denote the left twisted diagonal subgroup $_{(P,\varphi)}\Delta$. [We write the subgroup on the right to remind us that that the source of φ is thought of as living in the second copy of S.] Moreover, we write $[\varphi, P]$ for the transitive (S,S)-biset that contains a point with stabilizer (φ, P).

Finally, given an (S,S)-biset Ω and point $\omega \in \Omega$, if $\mathrm{Stab}(\omega) = (\varphi, P)$ we write $S_\omega := P$ and $c_\omega := \varphi$. Here we think of ω as acting by "conjugation" (according to the map φ) on the subgroup $S_\omega \leq S$, which is the largest subgroup of S "conjugated" into S by ω.

Assumption 8. For all $\omega \in \Omega$, $c_\omega \in \mathcal{F}(S_\omega, S)$.

We give a name to this condition:

Definition 9. Ω is *$\mathcal{F}$-generated* if every conjugation morphism of every element of Ω is a morphism of $\mathcal{F}$.

Remark 10. Note that $_SS_S \cong [\mathrm{id}_S, S]$, with the identity element having point-stabilizer (id_S, S). Thus we could rephrase Assumption 2 as saying that we require $[\mathrm{id}_S, S] \subseteq \Omega$.

The following are straightforward applications of the definitions:

Lemma 11. *If $\omega \in \Omega$ has point-stabilizer (c_ω, S_ω), then for all $a, b \in S$ we have*

$$\mathrm{Stab}(a \cdot \omega \cdot b) = (c_a \circ c_\omega \circ c_b, S_\omega^b).$$

Lemma 12. *If $[\varphi, P]$ is an (S,S)-orbit, then $[\varphi, P]^{\mathrm{op}} = [\varphi^{-1}, \varphi P]$.*

Next, we consider restrictions of the *left* action on Ω to subgroups of S. Suppose we have $g \in G$, and consider the effect of left multiplication by ℓ_g on $_SG_S$. For $S_g = S \cap S^g \leq S$, there are two (S_g, S)-biset structures on $_SG_S$ we might consider: The first is the restriction of the standard (S,S)-biset structure to S_g on the left, and the second is given by

$$a \odot g' = c_g(a) \cdot g',$$

i.e., $\odot$ is the restriction of $\cdot$ twisted by the fusion morphism c_g. We denote these (S_g, S)-bisets ${}_{S_g}G_S$ and ${}^{c_g}_{S_g}G_S$, respectively.

For all $a \in S_g$ and $g' \in G$ we have

$$\ell_g(a \cdot g') = ga \cdot g' = gag^{-1} \cdot g \cdot g' = c_g(a) \cdot \ell_g(g') = a \odot \ell_g(g').$$

Thus ℓ_g is an isomorphism of (S_g, S)-bisets ${}_{S_g}G_S \cong {}^{c_g}_{S_g}G_S$. This brings us to our final condition:

Assumption 13. For all $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$, ${}_P\Omega_S \cong {}^{\varphi}_{P}\Omega_S$ as (P,S)-bisets.

If we view (S,S)-bisets as $S \times S$-sets, we can apply the standard theory of marks to reformulate what it means for ${}_P\Omega_S$ and ${}^{\varphi}_{P}\Omega_S$ to be isomorphic. Recall that for H a finite group and X, Y finite left H-sets, $X \cong Y$ as H-sets if and only if for all $K \leq H$, the orders of the K-fixed points of X and the K-fixed points of Y are equal:

$$X \cong Y \qquad \Longleftrightarrow \qquad |X^K| = |Y^K| \ \forall\ K \leq H.$$

In the case of our biset Ω, viewed as an $S \times S$-set, we note that for any $K \leq S \times S$ we have $\omega \in \Omega^K$ iff $K \leq \mathrm{Stab}(\omega) = (c_\omega, S_\omega)$. Thus $\Omega^K = \emptyset$ unless K is a twisted diagonal subgroup of the form (φ, P) with $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$.

Definition 14.

- Ω is *left $\mathcal{F}$-stable* if $|\Omega^{(\varphi,P)}| = |\Omega^{(\iota,P)}|$ for all $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$.
- Ω is *right $\mathcal{F}$-stable* if $|\Omega^{(\iota,P)}| = |\Omega^{(\varphi^{-1},\varphi P)}|$ for all $P \leq S$ and $\varphi \in \mathcal{F}(P,S)$.
- Ω is *$\mathcal{F}$-stable* if it is left and right $\mathcal{F}$-stable.

Thus we can reformulate the last assumption as follows:

Assumption 15. Ω is $\mathcal{F}$-stable.

Note that the symmetry of Ω and the fact that morphism of $\mathcal{F}$ factors as an isomorphism followed by an inclusion imply that left $\mathcal{F}$-stability of Ω implies $\mathcal{F}$-stability.

Finally, note that $[G : S] = |G|/|S|$ is prime to p.

Assumption 16. $|\Omega/S|$ is prime to p.

We summarize our desired properties:

Definition 17. Let Ω be a (S,S)-biset and $\mathcal{F}$ a fusion system on S.

Ω is *$\mathcal{F}$-semicharacteristic* if Ω satisfies:

(0) Ω is bifree and symmetric.
(1) Ω is $\mathcal{F}$-generated.
(2) Ω is $\mathcal{F}$-stable.

Ω is *$\mathcal{F}$-characteristic* if it satisfies items (0)-(2) and additionally:

(3) $|\Omega/S|$ is prime to p.

Our goal is to show that if $\mathcal{F}$ is saturated then there exists an $\mathcal{F}$-characteristic biset. We will do more and explicitly parameterize all $\mathcal{F}$-semicharacteristic bisets. For notational convenience, this will require a short detour.

$\mathcal{F}$-sets

For this section we will consider left S-sets instead of (S,S)-bisets. The connection comes from the observation that an (S,S)-biset can be thought of as an $S \times S$-set, and the fusion system $\mathcal{F}$ on S naturally induces a fusion system $\mathcal{F} \times \mathcal{F}$ on $S \times S$, which is generated by morphisms defined on product subgroups $P \times P' \le S \times S$. We will explain this point further in the next section.

For now, we want to consider which S-sets might be thought of as being "acted on" by $\mathcal{F}$. It turns out that the most obvious definition is surprisingly useful:

Definition 18. The S-set X is *$\mathcal{F}$-stable*, or an $\mathcal{F}$-set, if for all $P \le S$ and $\varphi \in \mathcal{F}(P,S)$, there is an isomorphism of P-sets ${}_P X \cong {}^{\varphi}_{P} X$, or equivalently if $|X^P| = |X^{\varphi P}|$.

Note that the product and disjoint union of $\mathcal{F}$-sets are $\mathcal{F}$-sets, so the collection of all S-isomorphism classes of $\mathcal{F}$-sets form a monoid. In fact, this monoid is free with a natural basis indexed by the $\mathcal{F}$-isomorphism classes of subgroups of S. More precisely:

Theorem 19. *For each $\mathcal{F}$-conjugacy class $[P]$ of subgroups of S with fully $\mathcal{F}$-normalized representative P, there is a unique $\mathcal{F}$-set*

$$\alpha_P = \coprod_{Q \in \mathrm{cl}(S)} c_Q(\alpha_P) \cdot [S/Q]$$

that satisfies:

(i) $c_P(\alpha_P) = 1$,
(ii) *If $c_Q(\alpha_P) \neq 0$, then Q is $\mathcal{F}$-subconjugate to P,*
(iii) *If $c_Q(\alpha_P) \neq 0$ and Q is fully $\mathcal{F}$-normalized, then $Q \cong_{\mathcal{F}} P$.*

Moreover, these $\{\alpha_P\}$ form an additive basis for the monoid of $\mathcal{F}$-sets.

Recall that one of the key technical facts about saturated fusion systems is that if $Q \cong_{\mathcal{F}} P$ with P fully $\mathcal{F}$-normalized, then there is a morphism $\varphi \in \mathcal{F}(N_S Q, N_S P)$ such that $\varphi Q = P$. The following is the application of this result to fixed-point sets:

Lemma 20 (Reeh)**.** *Let X be an S set, $Q, P \le$ $\mathcal{F}$-conjugate subgroups such that P is fully $\mathcal{F}$-normalized. Suppose that every point-stabilizer of X has order greater than $|P|$. If for all subgroups $R \cong R'$ of order greater than $|P|$ we have $|X^R| = |X^{R'}|$, then $|X^Q| \le |X^P|$.*

Proof. I first claim that

$$X^Q = \bigcup_{Q \lneq R \le N_S Q} X^R.$$

The containment $\supseteq$ is obvious from the inclusion-reversing nature of fixed points. For the reverse containment, for every $x \in X^Q$ we must have $Q \lneq \mathrm{Stab}(x)$ by the assumption that all point-stabilizers have order greater than $|P| = |Q|$. Therefore $Q \lneq N_{\mathrm{Stab}(x)}(Q) \le N_S Q$, and x is a fixed point of $N_{\mathrm{Stab}(x)}(Q)$, so the equality is proved. Similarly,

$$X^P = \bigcup_{P \lneq R \le N_S P} X^R.$$

Now fix some $\varphi \in \mathcal{F}(N_S Q, N_S P)$ such that $\varphi Q = P$. By the assumption on equality of fixed-point orders for groups of order greater than $|P|$, we have $|X^R| = |X^{\varphi R}|$ for all $Q \lneq R \le N_S P$. An easy application of inclusion-exclusion then yields

$$|X^Q| = \left| \bigcup_{Q \lneq R \le N_S Q} X^R \right| = \left| \bigcup_{P \lneq R \le \varphi N_S Q} X^R \right| \le \left| \bigcup_{P \lneq R \le N_S P} X^R \right| = |X^P|,$$

and the result is proved. ∎

Proof of Theorem 19. Let $P \leq S$ be fully $\mathcal{F}$-normalized, and consider the S-set $[S/P]$. We proceed by downward induction on the subgroups of S, adding S-orbits $[S/Q]$ whenever the $\mathcal{F}$-stability condition (stated in terms of fixed points) fails. The point is that there is a unique minimal way to do this, thanks to Lemma 20.

First we must add orbits of the form $[S/Q]$ for $Q \cong_{\mathcal{F}} P$. The number of P-fixed points of $[S/P]$ is $|W_S P|$, while if $Q \cong_{\mathcal{F}} P$ but $Q \not\cong_S P$, $[S/P]^Q = \emptyset$. We can add $\frac{|W_S P|}{|W_S Q|} = \frac{|N_S P|}{|N_S Q|}$ copies of $[S/Q]$ to correct the inequality of fixed-point orders. Repeating this process for all $\mathcal{F}$-conjugates of P yields an S-set that satisfies the $\mathcal{F}$-stability condition for all subgroups of order at least $|P|$.

Suppose we now have $Q \cong_{\mathcal{F}} R$ with R fully $\mathcal{F}$-normalized and X satisfies the $\mathcal{F}$-stability condition for all groups of order greater than $|R|$. If $|X^Q| \neq |X^R|$, by Lemma 20 we must have $|X^Q| < |X^R|$. Since $|[S/Q]^Q| = |W_S Q|$, if we can show that $|W_S Q| \big| |X^R| - |X^Q|$, adding $\frac{|X^R|-|X^Q|}{|W_S Q|}$ copies of $[S/Q]$ will correct the $\mathcal{F}$-stability condition for Q and R without damaging the already established equality of fixed-point orders. We do this now.

Fix a morphism $\varphi \in \mathcal{F}(N_S Q, N_S R)$ such that $\varphi Q = R$. The Q-fixed points X^Q are naturally a $W_S Q$-set, and X^R becomes a $W_S Q$-set by restriction of the natural $N_S P$-action along φ. For Y either of the $W_S Q$-sets X^Q or X^R, we can partition $Y = Y_s \amalg Y_f$, where Y_s is the singular part (i.e., those $y \in Y$ such that $\mathrm{Stab}_{W_S Q}(y) \neq 1$) and Y_f is the free part of Y. We have

$$Y_s = \bigcup_{1 \neq T \leq W_S Q} Y^T,$$

and by the inductive assumption on X the T-fixed points of X^Q have the same order of the φT-fixed points on X^R. Thus $|X_s^Q| = |X_s^R|$, and $|X^R| - |X^Q| = |X_f^R| - |X_f^Q|$. But $|W_S Q|$ divides both $|X_f^R|$ and $|X_f^Q|$, so we are done with the construction of α_P.

To see that the $\{\alpha_P\}$ form a basis for the S-isomorphism classes of $\mathcal{F}$-sets, simply note that each α_P is characterized by having a unique S-orbit of the form $[S/P]$ for P fully $\mathcal{F}$-normalized. Thus if X is an arbitrary $\mathcal{F}$-set with $c_P(X)$ copies of $[S/P]$ for P fully $\mathcal{F}$-normalized, the element $X - \sum c_P(X) \cdot \alpha_P$ of the Burnside ring of S must have no fixed points for all $Q \leq S$. Thus $X = \sum c_P(X) \cdot \alpha_P$, and the Theorem is proved. □

Again, the main point of this result is that we can read off the decomposition of an X-set $\mathcal{F}$ in terms of the basis $\{\alpha_P\}$ by concentrating solely on the number of orbits of the form $[S/P]$ for P fully $\mathcal{F}$-normalized. An easy induction then yields:

Corollary 21. *If X is an S-set, there is a unique minimal $\mathcal{F}$-set $\widetilde{X}$ containing X such that every point-stabilizer of $\widetilde{X}$ is $\mathcal{F}$-subconjugate to a point-stabilizer of X.*

This minimal $\mathcal{F}$-set containing a given S-set X is called the *$\mathcal{F}$-stabilization of X*.

The parameterization of $\mathcal{F}$-semicharacteristic bisets

We return to our investigation of the semicharacteristic bisets of the saturated fusion system $\mathcal{F}$. As alluded to earlier, we view an (S,S)-biset as an $S \times S$-set. Once we place a fusion system $\mathcal{F} \times \mathcal{F}$ on $S \times S$, we can speak of $\mathcal{F} \times \mathcal{F}$-sets, which will turn out to be equivalent to the $\mathcal{F}$-stable (S,S)-bisets.

Definition 22. $\mathcal{F} \times \mathcal{F}$ is the fusion system on $S \times S$ generated by $\mathcal{F}$ in each coordinate. Explicitly, if $P \leq S \times S$ has projection $P_1 := \mathrm{pr}_1(P)$ onto the first coordinate and $P_2 := \mathrm{pr}_2(P)$ onto the second, we define $\mathcal{F} \times \mathcal{F}(P, S \times S)$ to be those maps that are restrictions of $\varphi_1 \times \varphi_2$ for $\varphi_i \in \mathcal{F}(P_i, S)$.

Lemma 23. *If $\mathcal{F}$ is saturated, then $\mathcal{F} \times \mathcal{F}$ is as well.*

Proof. We first claim that any P is $\mathcal{F} \times \mathcal{F}$-conjugate to P' such that P' is fully $\mathcal{F} \times \mathcal{F}$-centralized and $\operatorname{Aut}_{S\times S}(P') \in \operatorname{Syl}_p(\operatorname{Aut}_{\mathcal{F}\times\mathcal{F}}(P'))$. Let P_1, P_2 denote the projections of P onto the first and second coordinates; without loss of generality we may assume that each P_i is fully $\mathcal{F}$-normalized. Then $C_{S\times S}(P) = C_S(P_1) \times C_S(P_2)$, so each P_i is fully $\mathcal{F}$-centralized. Since the $S \times S$-centralizer of any subgroup is the product of the S-centralizers of its projections, this implies that P is fully $\mathcal{F} \times \mathcal{F}$-centralized as well. The saturation of $\mathcal{F}$ implies that $\operatorname{Aut}_S(P_i) \in \operatorname{Syl}_p(\operatorname{Aut}_{\mathcal{F}}(P_i))$, so it follows that

$$\operatorname{Aut}_{S\times S}(P_1 \times P_2) \in \operatorname{Syl}_p(\operatorname{Aut}_{\mathcal{F}\times\mathcal{F}}(P_1 \times P_2)).$$

We can view $\operatorname{Aut}_{\mathcal{F}\times\mathcal{F}}(P)$ as the subgroup of $\operatorname{Aut}_{\mathcal{F}\times\mathcal{F}}(P_1 \times P_2)$ that sends P to itself (because any automorphism of $P_1 \times P_2$ is determined by its action on P), so there is some $\alpha \in \operatorname{Aut}_{\mathcal{F}\times\mathcal{F}}(P_1 \times P_2)$ such that $\operatorname{Aut}_{S\times S}(P_1 \times P_2)$ contains a Sylow p-subgroup of ${}^{\alpha}\operatorname{Aut}_{\mathcal{F}\times\mathcal{F}}(P) = \operatorname{Aut}_{\mathcal{F}\times\mathcal{F}}(\alpha P)$. Then αP is still fully $\mathcal{F} \times \mathcal{F}$-centralized and

$$\operatorname{Aut}_{S\times S}(\alpha P) = \operatorname{Aut}_{S\times S}(P_1 \times P_2) \cap \operatorname{Aut}_{\mathcal{F}\times\mathcal{F}}(\alpha P) \in \operatorname{Syl}_p(\operatorname{Aut}_{\mathcal{F}\times\mathcal{F}}(\alpha P))$$

and the first claim is proved.

Note that this implies that being fully $\mathcal{F} \times \mathcal{F}$-normalized implies being fully $\mathcal{F} \times \mathcal{F}$-centralized and that the $S \times S$-automorphisms are Sylow in the $\mathcal{F} \times \mathcal{F}$-automorphisms. Indeed, for simplicity of notation let us assume that $\mathcal{G}$ is a fusion system on T such that every $P \le T$ is $\mathcal{G}$-conjugate to some P', fully $\mathcal{G}$-centralized and with $\operatorname{Aut}_{\mathcal{G}}(P') \in \operatorname{Syl}_p(\operatorname{Aut}_{\mathcal{G}}(P'))$. In particular, if P is fully $\mathcal{G}$-normalized, then $\operatorname{Aut}_S(P) = N_S P / C_S P$ is a p-subgroup of $\operatorname{Aut}_{\mathcal{G}}(P) \cong \operatorname{Aut}_{\mathcal{G}}(P')$, so that $|\operatorname{Aut}_S(P)| \le |\operatorname{Aut}_S(P')|$. By assumption that P is fully $\mathcal{G}$-normalized, we have $|N_S P| \ge |N_S P'|$, and by assumption that P' is fully $\mathcal{G}$-centralized we have $|C_S P| \le |C_S P'|$. Therefore we have

$$\frac{|N_S P|}{|C_S P|} = |\operatorname{Aut}_S(P)| \le |\operatorname{Aut}_S(P')| = \frac{|N_S P'|}{|C_S P'|}$$

but also $\frac{|N_S P|}{|C_S P|} \ge \frac{|N_S P'|}{|C_S P'|}$, so in fact we must have equality. This forces $|C_S P| = |C_S P'|$, so that P is fully $\mathcal{G}$-centralized, and $\operatorname{Aut}_S(P) \in \operatorname{Syl}_p(\operatorname{Aut}_{\mathcal{F}}(P))$ as well.

To prove the extension axiom, suppose that $P \le S \times S$ and $\varphi \in \mathcal{F} \times \mathcal{F}(P, S \times S)$ such that φP is fully $\mathcal{F} \times \mathcal{F}$-centralized. If the components of φ are φ_1 and φ_2, it is immediate that $N_\varphi \le N_{\varphi_1} \times N_{\varphi_2}$, so the extension axiom for $\mathcal{F}$ implies the same for $\mathcal{F} \times \mathcal{F}$. ☐

We are ready to describe a basis for the $\mathcal{F}$-semicharacteristic bisets. We now have that the monoid of $\mathcal{F} \times \mathcal{F}$-sets has a basis indexed by the $\mathcal{F} \times \mathcal{F}$-classes of subgroups of $S \times S$, and from our description of these basis elements (in particular that every point stabilizer of α_P is $\mathcal{F} \times \mathcal{F}$-subconjugate to P), we see that the basis elements corresponding to *bifree* bisets are indexed by the twisted diagonal subgroups $\alpha_{(\varphi,P)}$. Such a biset will be $\mathcal{F}$-generated iff $\varphi \in \mathcal{F}(P, S)$, in which case $(\varphi, P) \cong_{\mathcal{F}\times\mathcal{F}} (\iota, P)$. Furthermore:

Lemma 24. $|N_{S\times S}((\varphi, P))| = |N_\varphi| \cdot |C_S \varphi P|$.

Proof. Suppose that $(n, m) \in N_{S\times S}((\varphi, P))$. Thus for all $a \in P$ we have

$$(c_n(\varphi(a)), c_m(a)) = (\varphi(b), b) \in (\varphi, P),$$

for some $b \in P$. Then $b = c_m(a)$ so $\varphi \circ c_m(a) = c_n \circ \varphi(a)$, or $\varphi \circ c_m \circ \varphi^{-1} \in \operatorname{Aut}_S(P)$. Thus $m \in N_\varphi$ by definition. Conversely, given $m \in N_\varphi$, by definition there is some n that makes the above equality true. Thus the projection onto the second coordinate of $N_{S\times S}((\varphi, P))$ is precisely N_φ.

Once m is fixed, there are $|C_S\varphi P|$ choices of n that can appear in the first coordinate, and the result is proved. □

Corollary 25. *(ι, P) is fully $\mathcal{F}\times\mathcal{F}$-normalized iff P is fully $\mathcal{F}$-normalized.*

Definition 26. If P is fully $\mathcal{F}$-normalized, let Ω_P denote the $\mathcal{F}\times\mathcal{F}$-stabilization of (ι, P).

We have therefore proved that every $\mathcal{F}$-generated, $\mathcal{F}$-stable (S, S)-biset can be written uniquely as a sum of the $\{\Omega_P\}$, so these form a basis for the $\mathcal{F}$-semicharacteristic bisets. Finally, note that $|\Omega_P|/|S|$ is divisible by p if $P \lneq S$, while for Ω_S there are precisely $|\operatorname{Out}_{\mathcal{F}}(S)|$ orbits of order $|S|$. Since $\mathcal{F}$ is saturated, $p \nmid |\operatorname{Out}_{\mathcal{F}}(S)|$, and we have our main result:

Theorem 27. *An $\mathcal{F}$-semicharacteristic biset Ω can be written uniquely*

$$\Omega = \sum c_P(\Omega)\Omega_P,$$

and Ω is $\mathcal{F}$-characteristic if and only iff $p \nmid c_S$. In particular, $\mathcal{F}$ has $\mathcal{F}$-characteristic bisets, and Ω_S is the unique minimal such.