Fusion systems seminar 2:
Toward classifying spaces for fusion systems

Speaker: Matthew Gelvin
Date: 13/10/15

Goals:

(i) Introduce the notion of isomorphic fusion systems
(ii) Describe (without proof) the relevance of fusion systems to algebraic topology
(iii) Introduce the orbit system associated to a fusion system

Let p be a prime and G a finite group with $p \big| |G|$.

Groups with isomorphic fusion

We will be concerned with realizable fusion systems in this section; that is, we assume that $\mathcal{F} = \mathcal{F}_S(G)$ for some $S \in \mathrm{Syl}_p(G)$. Clearly $\mathcal{F}$ contains less information than G, and in fact many G can give rise to the same fusion system. In this simplest case where H is your favorite p'-group, then $\mathcal{F}_S(G \times H) = \mathcal{F}_S(G)$, but more complicated situations may arise as well (e.g., $\mathcal{F}_{D_8}(\Sigma_4) = \mathcal{F}_{D_8}(\Sigma_5)$). We say that all such groups have the same p-local data, as expressed by having isomorphic fusion systems. We formalize this notion as follows:

Definition 1. Let $(S_1, \mathcal{F}_1)$ and $(S_2, \mathcal{F}_2)$ be fusion systems and $\gamma : S_1 \to S_2$ be a p-group isomorphism. γ is a $(\mathcal{F}_1, \mathcal{F}_2)$*-fusion-preserving isomorphism* if for all $P, Q \leq S$, the map

$$\gamma_* : \mathrm{Hom}(P, Q) \to \mathrm{Hom}(\gamma(P), \gamma(Q)) : \varphi \mapsto \gamma\varphi\gamma^{-1}$$

restricts to a bijection $\mathcal{F}_1(P, Q) \cong \mathcal{F}_2(\gamma(P), \gamma(Q))$.

If there exists a $(\mathcal{F}_1, \mathcal{F}_2)$-fusion-preserving isomorphism from S_1 and S_2, we say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are *ismorphic* and write $\mathcal{F}_1 \cong \mathcal{F}_2$.

There is a categorical condition that determines when two fusion systems are isomorphic.

Definition 2. Let $\mathtt{pGrp} \subseteq \mathtt{Grp}$ denote the category of finite p-groups. For G a finite group, the *G-representation presheaf* is the functor $Rep_G : \mathtt{pGrp}^{\mathrm{op}} \to \mathtt{Set}$ defined by $R_G(P) = Rep_G(P, G) = \mathrm{Inn}(G)\backslash \mathrm{Hom}(P, G)$.

Proposition 3. *If $\mathcal{F}_i = \mathcal{F}_{S_i}(G_i)$ for $i = 1, 2$, then $\mathcal{F}_1 \cong \mathcal{F}_2$ iff $Rep_{G_1} \cong Rep_{G_2}$ as presheaves on* $\mathtt{pGrp}$.

Proof. If $\gamma : S_1 \to S_2$ is a $(\mathcal{F}_1, \mathcal{F}_2)$-preserving isomorphism, we define a natural transformation $\eta : \mathrm{Rep}_{G_1} \Rightarrow \mathrm{Rep}_{G_2}$ as follows: For $P \in \mathtt{pGrp}$ and $[\delta] \in \mathrm{Rep}_{G_1}(P, G_1)$, we may assume by Sylow's theorem that the representative group map δ of the representation $[\delta]$ has image contained in S_1. Set $\eta_P([\delta]) = [\gamma\delta]$. This is well-defined: If $\widetilde{\delta} \in [\delta]$ also has image contained in S_1, then $\widetilde{\delta} = c_g\delta$ for some $g \in G_1$, so $c_g \in \mathcal{F}_1(\delta(P), \widetilde{\delta}(P))_{\mathrm{iso}}$. As γ is fusion-preserving, this implies that $\gamma c_g \gamma^{-1} \in \mathcal{F}_2(\delta(P), \gamma\widetilde{\delta}(P))_{\mathrm{iso}}$, so there is some $g' \in G_2$ such that $c_{g'} = \gamma c_g \gamma^{-1}$, or $c_{g'}\gamma = \gamma c_g : \delta(P) \to G_2$. Thus

$$[\gamma c_g \delta] = [c_{g'}\gamma\delta] = [\gamma\delta],$$

so that η_P does not depend on the choice of representative of $[\delta]$ (so long as the image is contained in S_1). As the Rep_{G_i} act on morphisms by precomposition, and η acts by postcomposition, it is immediate that η is natural. The construction of η^{-1} is immediate from the symmetry f the above definition.

Conversely, suppose we are given $\eta : Rep_{G_1} \cong Rep_{G_2}$. I claim that η must send injective representations to injective representations. Suppose we have an map $\delta : P \hookrightarrow G_1$ such that $\eta_P([\delta]) = [\zeta]$ where $\zeta \in \mathrm{Hom}(P, G_2)$ has kernel Q. Let $\pi : P \to P/Q$ be the natural projection, and consider the commuting diagram

$$\begin{array}{ccccc} Rep_{G_1}(P) & \xrightarrow{\eta_P} & Rep_{G_2}(P) & \xrightarrow{\eta_P^{-1}} & Rep_{G_1}(P) \\ \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} \\ Rep_{G_1}(P/Q) & \xrightarrow[\eta_{P/Q}]{} & Rep_{G_2}(P/Q) & \xrightarrow[\eta_{P/Q}^{-1}]{} & Rep_{G_1}(P/Q) \end{array}$$

If $\overline{\delta} \in \mathrm{Hom}(P/Q, G_2)$ is the map induced by δ, we have $\eta_{P/Q}^{-1}([\overline{\delta}]) \in \mathrm{Rep}_{G_1}(P/Q, G_1)$ maps to $[\delta]$ via π^*. But this means that $c_g\delta$ factors through P/Q for some $g \in G_1$, so that δ must have kernel Q as well.

Now for the inclusion $\iota_{S_1}^{G_1}$, we have $\eta_{S_1}([\iota_{S_1}^{G_1}]) = [\gamma] \in \mathrm{Rep}_{G_2}(S_1, G_2)$, and we may choose a representative γ with $\gamma(S_1) \leq S_2$. By the above, γ is an injection, so that $|S_1| \leq |S_2|$. The dual argument for $\iota_{S_2}^{G_2}$ implies that $|S_2| \leq |S_1|$, so in fact γ is an isomorphism.

I claim that γ is an $(\mathcal{F}_1, \mathcal{F}_2)$-preserving isomorphism. As every morphism in a fusion system factors as an isomorphism followed by an inclusion, it suffices to show that for $P \leq S_1$ and $\varphi \in \mathrm{Hom}(P, S_1)$, we have $\varphi \in \mathcal{F}_1(P, S_1)$ iff $\gamma\varphi\gamma^{-1} \in \mathcal{F}_2(\gamma P, S_2)$. Consider the following commuting diagrams:

$$\begin{array}{ccc} \mathrm{Rep}_{G_1}(S_1, G_1) & \xrightarrow{\eta_{S_1}} & \mathrm{Rep}_{G_2}(S_1, G_2) \\ {\scriptstyle \varphi^*}\downarrow & & \downarrow{\scriptstyle \varphi^*} \\ \mathrm{Rep}_{G_1}(P, G_1) & \xrightarrow{\eta_P} & \mathrm{Rep}_{G_2}(P, G_2) \\ {\scriptstyle (\gamma^{-1})^*}\downarrow & & \downarrow{\scriptstyle (\gamma^{-1})^*} \\ \mathrm{Rep}_{G_1}(\gamma P, G_1) & \xrightarrow{\eta_{\gamma P}} & \mathrm{Rep}_{G_2}(\gamma P, G_2) \end{array} \qquad \begin{array}{ccccc} [\iota_{S_1}^{G_1}] & \longmapsto & [\gamma] & & \\ \downarrow & & \downarrow & & \\ [\varphi] & \longmapsto & \eta_P([\varphi]) & = & [\varphi\gamma] \\ \downarrow & & \downarrow & & \\ [\varphi\gamma^{-1}] & \longmapsto & \eta_{\gamma P}([\varphi\gamma^{-1}]) & = & [\gamma\varphi\gamma^{-1}] \end{array}$$

$$\begin{array}{ccc} \mathrm{Rep}_{G_1}(S_1, G_1) & \xrightarrow{\eta_{S_1}} & \mathrm{Rep}_{G_2}(S_1, G_2) \\ {\scriptstyle (\iota_P^{S_1})^*}\downarrow & & \downarrow{\scriptstyle (\iota_P^{S_1})^*} \\ \mathrm{Rep}_{G_1}(P, G_1) & \xrightarrow{\eta_P} & \mathrm{Rep}_{G_2}(P, G_2) \\ {\scriptstyle (\gamma^{-1})^*}\downarrow & & \downarrow{\scriptstyle (\gamma^{-1})^*} \\ \mathrm{Rep}_{G_1}(\gamma P, G_1) & \xrightarrow{\eta_{\gamma P}} & \mathrm{Rep}_{G_2}(\gamma P, G_2) \end{array} \qquad \begin{array}{ccccc} [\iota_{S_1}^{G_1}] & \longmapsto & [\gamma] & & \\ \downarrow & & \downarrow & & \\ [\iota_P^{G_1}] & \longmapsto & \eta_P([\iota_P^{G_1}]) & = & [\gamma] \\ \downarrow & & \downarrow & & \\ [\gamma^{-1}] & \longmapsto & \eta_{\gamma P}([\gamma^{-1}]) & = & [\iota_{\gamma P}^{G_2}] \end{array}$$

Thus $\varphi \in \mathcal{F}_1(P, S_1)$ iff $[\varphi] = [\iota_P^{G_1}] \in \mathrm{Rep}_{G_1}(P, G_1)$ iff $[\gamma\varphi\gamma^{-1}] = [\iota_{\gamma P}^{G_2}] \in \mathrm{Rep}_{G_2}(\gamma P, G_2)$ iff $\gamma\varphi\gamma^{-1} \in \mathcal{F}_2(\gamma P, S_2)$, and the result is proved. □

The Martino-Priddy conjecture

The purpose of this section is to describe in broad strokes the role of fusion systems in algebraic topology. As this is mostly intended for the purpose of motivation, and we will be relying on some heavy machinery, we will not prove everything in detail. Instead, we will concentrate on building the framework that led topologists to be interested in orbit and linking systems in the first place.

We begin with the homotopy-theoretic version of a finite group:

Definition 4. Let $\mathcal{E}G$ be the category with objects $\{g | g \in G\}$ and every hom-set containing a single morphism. Let $EG = |\mathcal{E}G|$ be the geometric realization of $\mathcal{E}G$.

Let $\mathcal{B}G$ be the category with a single object $*$ and $\mathcal{B}G(*) = G$. Let $BG = |\mathcal{B}G|$ be the *classifying space* of G.

A few points to note:

- $EG \simeq *$ and EG is a free G-space.
- There is a G-quotient map $EG \to BG$, which is a finite cover.
- Therefore BG is a $K(G,1)$, i.e., $\pi_i(BG) = \begin{cases} G & i = 1, \\ 0 & \text{else.} \end{cases}$

Some basic simplicial set theory yields the following:

Proposition 5. *Let G and H be finite groups with classifying spaces BG and BH. Then*

- $[BG, BH]_* \cong \mathrm{Hom}(G,H)$, *and*
- $[BG, BH] \cong \mathrm{Rep}(G,H) := \mathrm{Inn}(H)\backslash \mathrm{Hom}(G,H)$,

where $[X,Y]_*$ is the set of pointed homotopy classes of maps from X to Y, and $[X,Y]$ is the set of unpointed homotopy classes of maps.

Thus one can translate between the worlds of group theory and algebraic topology by means of the classifying space. [Aside: Under this dictionary, a G-set is precisely a functor $\mathcal{B}G \to \mathtt{Set}$, and by considering the Grothendieck category of such a functor one sees that the theory of G-sets corresponds precisely to the covering spaces of BG.] In light of the previous section, that $\mathrm{Rep}(P,G)$ can be described in homotopical terms hints at a topological interpretation of two groups having the same p-local data.

Definition 6. If $f : X \to Y$ is a map of spaces, f is *p-equivalence* if the induced homology map $\mathrm{H}_*(X;\mathbb{Z}/p) \to \mathrm{H}_*(Y;\mathbb{Z}/p)$ is an isomorphism, or equivalently if the induced cohomology map $\mathrm{H}^*(Y;\mathbb{Z}/p) \to \mathrm{H}^*(X;\mathbb{Z}/p)$ is an isomorphism.

If there exists a p-equivalence from $f : X \to Y$, we say X and Y are *p-equivalent* and write $X \simeq_p Y$.

For the rest of these notes, homology and cohomology will always be considered with $\mathbb{Z}/p$-coefficients, so we will write $\mathrm{H}_*(X)$ for $\mathrm{H}_*(X;\mathbb{Z}/p)$, etc.

When presented with a new notion of equivalence, homotopy theorists love nothing more than to imagine a world in which those equivalences are actually invertible morphisms. Enter the Bousfield-Kan p-completion functor:

Definition 7. *Bousfield-Kan p-completion* is a functor $(-)^\wedge_p : \mathtt{Spc} \to \mathtt{Spc}$ together with a natural transformation $\eta : \mathrm{Id}_{\mathtt{Spc}} \to (-)^\wedge_p$. The p-completion functor is characterized by the condition that $X \xrightarrow{f} Y$ is a p-equivalence iff $X^\wedge_p \xrightarrow{f^\wedge_p} Y^\wedge_p$ is a homotopy equivalence.

Here, $\mathtt{Spc}$ is the category of spaces, aka simplicial sets. For our purposes, we can think of this as a category of well-behaved topological spaces, e.g. CW-complexes.

Definition 8. Let X be a space.

- X is *p-complete* if $\eta_X : X \to X^\wedge_p$ is a homotopy equivalence.
- X is *p-good* if $\eta_X : X \to X^\wedge_p$ is a p-equivalence.

Note that that X is p-good iff $X_p^\wedge$ is p-complete, and the category of p-complete spaces is precisely the subcategory of spaces in which p-equivalences are (homotopy) invertible.

We are not going to construct the p-completion functor here or describe its operations in any greater detail. We will however note the following fundamental results:

- A space is either p-good or p-very, very bad: If $\eta_X : X \to X_p^\wedge$ is not a p-equivalence, no number of repeated applications of the p-completion functor will make it one.
- If X is p-good, $\eta_X : X \to X_p^\wedge$ is terminal among all p-equivalences with source X.
- If X and Y are spaces with at least one p-good, $X \simeq_p Y$ iff there is a space Z and p-equivalences $X \to Z \leftarrow Y$.
- If $\pi_1(X)$ is finite, then X is p-good. In this case, $\pi_1(X_p^\wedge) \cong \pi_1(X)/O^p(\pi_1(X))$, where $O^p(G)$ denotes the minimal normal p-power index subgroup of G.
- If P is a finite p-group, then BP is p-good.

The importance of p-completion is further revealed by the following theorem of Mislin:

Theorem 9. *If P is a p-group and G is a finite group, p-completion of BG induces a bijection $[BP, BG] \to [BP, BG_p^\wedge]$.*

In particular, we can compute $\text{Rep}(P, G) \cong [BP, BG_p^\wedge]$. This result is very deep and uses machinery involved with the solution of the Sullivan Conjecture, which we will not discuss further. Note however that, in light of the preceding section, this implies:

Corollary 10. *If $BG \cong_p BH$, G and H have the same p-local data.*

This is half of the bridge between algebra and topology, known as the Martino-Priddy Conjecture but which has been proved by Oliver:

Theorem 11 (Martino-Priddy Conjecture)**.** *Let G and H be finite groups with Sylow p-subgroups S and T, respectively. Then $\mathcal{F}_S(G) \cong \mathcal{F}_T(H)$ iff $BG \simeq_p BH$.*

This says that the topological and algebraic notions of the p-local structure of finite groups coincide. We've already seen the "easy" implication from topology to algebra. It seems reasonable that this should be comparatively straightforward, as there is (on the face of it) more structure in the p-complete classifying space $BG_p^\wedge$ than in the fusion system $\mathcal{F}_S(G)$. Reversing the implication will require some novel ideas.

The search for $B\mathcal{F}$

Deriving topological data from algebraic will require, on some level, the construction of a classifying space associated to a fusion system $\mathcal{F}$. To prove the Martino-Priddy conjecture, in the case that $\mathcal{F} = \mathcal{F}_S(G)$ we need for this space to be p-equivalent to BG. Therefore, we are looking for a space $B\mathcal{F}$ that (ideally) satisfies:

(i) $B\mathcal{F}$ is p-good.
(ii) If $\mathcal{F} = \mathcal{F}_S(G)$, $B\mathcal{F} \simeq_p BG$.
(iii) There is a natural map $\beta : BS \to B\mathcal{F}$.
(iv) $\mathcal{F}$ can be recovered from β.
(v) β is *Sylow*: For any finite group P and map $f : BP \to B\mathcal{F}$, f factors through β.
(vi) β is a *homotopy monomorphism at p*: $H^*(BS)$ is a finitely generated $H^*(B\mathcal{F})$-module via the induced map on cohomology $H^*(B\mathcal{F}) \to H^*(BS)$.
(vii) There is a stable transfer map $t : \Sigma_+^\infty B\mathcal{F} \to \Sigma_+^\infty BS$ with $\Sigma_+^\infty \beta \circ t \cong \text{id}_{B\mathcal{F}}$ and certain reciprocity conditions. In particular, $B\mathcal{F}$ is a stable summand of BS.
(viii) The assignment $\mathcal{F} \to B\mathcal{F}$ is functorial.

We will only concentrate on items (i) through (v). Items (vi) and (vii) are true, and are motivated by transfer theory for Sylow subgroups of finite groups (in particular, that in group cohomology induction followed by restriction is multiplication by the group index) and Ragnarsson's work on *retractive transfer triples*, but we will not touch on them here. Item (viii) is still a dream.

So, how can we make a space from a fusion system? A first attempt is to view $\mathcal{F}$ as a diagram in $\mathtt{pGrp}$, so applying the classifying space functor $B(-)$ yields a diagram in $\mathtt{Spc}$. We then "glue" the subgroups of S together according to $\mathcal{F}$ via the homotopy colimit construction, a homotopically invariant analogue of the colimit of spaces:

$$B\mathcal{F} \overset{?}{=} \operatorname*{hocolim}_{P \in \mathcal{F}} BP.$$

Unfortunately, right away we run into problems. To understand the issue, we need a good model for the homotopy colimit.

Definition 12. Let $F : I \to \mathtt{Cat}$ be a diagram in small categories. The *Grothendieck construction* of F is the category $\mathfrak{G}(F)$, which has as objects $\{(i, c) | i \in I, c \in \mathrm{ob}(F(i))\}$, and the hom-set from (i, c) to (i', c') is the set of all pairs

$$\{(\gamma, \phi) | \gamma \in I(c, c'), \phi \in F(i')(F(\gamma)(c), c')\}.$$

As ungainly as the Grothendieck construction may appear at first, it provides a useful combinatorial model for the homotopy colimit:

Theorem 13 (Thomasson)**.** *If $F : I \to \mathtt{Cat}$ is a diagram in small categories, then*

$$\operatorname*{hocolim}_{i \in I} |F(i)| \cong |\mathfrak{G}(F)|.$$

We can view the functor $\mathcal{F} \to \mathtt{Spc} : P \mapsto BP$ as factoring through the classifying category functor $\mathcal{B} : \mathcal{F} \to \mathtt{Cat} : P \mapsto \mathcal{B}P$. Thus, the proposed space $B\mathcal{F}$ can be seen as the realization of a particular category, which is not particularly difficult to describe.

As $\mathcal{B}P$ is a one-object catgory, we can think of the objects of $\mathfrak{G}(\mathcal{B})$ as simply the set of subgroups of S. A morphism from P to Q is then a pair (φ, q), where $\varphi \in \mathcal{F}(P, Q)$ and $q \in Q$, and composition is defined by

$$P \xrightarrow{(\varphi,q)} Q \xrightarrow{(\psi,r)} R, \qquad P \xrightarrow{(\psi\varphi,\,\psi(q)\cdot r)} R$$

This formula might look somewhat familiar: If G, H are groups and $\phi : G \to \mathrm{Aut}(H)$ is a group-map, the semidirect product $G \ltimes_\phi H$ is defined as a set to be $G \times H$ with multiplication $(g, h) \cdot (g', h') = (gg', \phi(g')(h) \cdot h')$. We might therefore think of the Grothendieck construction as a generalization of the semidirect product from groups to categories. This is a useful heuristic, which also shows why our initial attempt at finding $B\mathcal{F}$ fails.

Suppose that S is a nonabelian group with fusion system $\mathcal{F}$. Then there is a functor $\mathcal{B}S \to \mathfrak{G}(\mathcal{B})$ that sends the single object of $\mathcal{B}S$ to the object $S \in \mathfrak{G}(\mathcal{B})$, and sends the morphism $x \in S$ to (id, s). This induces the map $BS \to \operatorname*{hocolim}_{P \in \mathcal{F}} \mathcal{B}P$ that we desire, but it is not Sylow: By construction, $\mathrm{Inn}(S) \ltimes S \leq \mathrm{Aut}_{\mathfrak{G}(\mathcal{B})}(P)$, so there is a functor from $\mathcal{B}(\mathrm{Inn}(S) \ltimes S)$ to the colimit, which does not factor through our original map.

The problem is that noncentral elements are double-counted in $\mathfrak{G}(\mathcal{B})$: Once as an element of S, and again as a nontrivial automorphism of S. [Incidentally, if S is abelian, this problem goes away and we do recover the correct classifying space $B\mathcal{F}$. This follows from Axiom (Ext) of saturation.] Let us therefore kill the offending, double-counted automorphisms.

Definition 14. The *orbit system associated to* $\mathcal{F}$ is the category $\mathcal{O}(\mathcal{F})$ whose objects are the subgroups of S and whose hom-sets are defined by $\mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P,Q) = \mathrm{Inn}(Q)\backslash\mathcal{F}(P,Q)$.

Definition 15. For G a finite group, the *orbit category of* G, $\Gamma(G)$, is the category whose objects are the subgroups of G and whose hom-sets are given by

$$\mathrm{Hom}_{\Gamma(G)}(H,K) = \mathrm{Hom}_G(G/H, G/K) \cong K\backslash N_G(H,K).$$

That is, $\Gamma(G)$ is equivalent to the category of transitive G-sets and G-maps.

The *p-orbit category of* G is the fully subcategory $\Gamma_p(G) \subseteq \Gamma(G)$ whose objects are the p-subgroups of G.

These notions are related, but the relationship is quite subtle: If $\mathcal{F} = \mathcal{F}_S(G)$, we can identify $\mathcal{F}(P,Q) \cong N_G(P,Q)/C_G(P)$, so $\mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P,Q) \cong Q\backslash N_G(P,Q)/C_G(P)$. Thus, up to equivalence of categories, we can view $\mathcal{O}(\mathcal{F})$ as a quotient of $\Gamma_p(G)$. Unfortunately, many nice properties of $\Gamma_p(G)$ are lost along the way, most notably that every morphism of $\Gamma_p(G)$ is categorically epi. We will return to the importance of this point at a later date.

The functor $B : \mathcal{F} \to \mathtt{Spc}$ does not induce a functor on $\mathcal{O}(\mathcal{F})$. However, as c_g induces a map homotopic to $\mathrm{id} : BG \to BG$ for all $g \in G$, we do have a well-defined functor into the homotopy category $\overline{B} : \mathcal{O}(\mathcal{F}) \to \mathtt{hoSpc}$. The problem now is that we need an actual diagram in $\mathtt{Spc}$ to form the homotopy colimit of the diagram; diagrams that commute up to homotopy are too flabby for the homotopy colimit to be well-defined. We thus are reduced to finding a *rigidification* $\widetilde{B} : \mathcal{O}(\mathcal{F}) \to \mathtt{Spc}$ of $\overline{B}$, i.e. a factorization

$$\begin{array}{ccc} & & \mathtt{Spc} \\ & \overset{\widetilde{B}}{\nearrow} & \downarrow \pi \\ \mathcal{O}(\mathcal{F}) & \xrightarrow[\overline{B}]{} & \mathtt{hoSpc} \end{array}$$

where π is the natural quotient functor to the homotopy category. Finding this rigidification will require our last major new idea. . . .

Which will be a topic for another day.