Fusion systems seminar 1:
A gentle introduction to fusion systems

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Goals:

(i) Introduce fusion systems (of finite groups and their abstraction)
(ii) Prove a key local-to-global structure theorem of fusion systems

Let p be a prime and G a finite group with $p \big| |G|$.

Fusion systems of groups

We are interested in the p-local structure of G. Our starting point will be Sylow's Theorem, which says

- Maximal order p-subgroups of G exist, eponymously called *Sylow subgroups*
- All Sylow subgroups are G-conjugate
- Every p-subgroup of G is contained in some Sylow subgroup
- Something about the number of Sylow subgroups, which we won't care about

This suggests that we should think of the "p-local structure of G" as information about the p-subgroups of G, together with the data of how they are identified via G-conjugacy (or *fusion*).

Definition 1. The *p-fusion system of G* is the category $\mathcal{F}_p(G)$ with objects the p-subgroups of G and morphisms given by

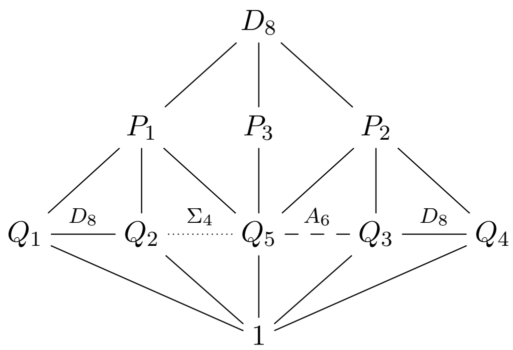
$$\mathrm{Hom}_{\mathcal{F}_p(G)}(P,Q) = \mathrm{Hom}_G(P,Q) = \{\varphi : P \to Q | \exists g \in G \text{ s.th. } \varphi = c_g|_P\} \cong N_G(P,Q)/C_G(P)$$

where $N_G(P,Q) := \{g \in G | {}^gP \leq G\}$ is the *G-transporter from P to Q*.

For $S \in \mathrm{Syl}_p(G)$, the *G-fusion system on S* is the full subcategory $\mathcal{F}_S(G) \subseteq \mathcal{F}_p(G)$.

An immediate consequence of Sylow's Theorem is that the inclusion $\mathcal{F}_S(G) \subseteq \mathcal{F}_p(G)$ is an equivalence of categories.

Example 2. Consider the 2-group D_8, which is Sylow in itself, Σ_4, and A_6. Note that $\mathcal{F}_{D_8}(D_8) \subset \mathcal{F}_{D_8}(\Sigma_4) \subset \mathcal{F}_{D_8}(A_6)$. We can visualize some of the data of these fusion systems by considering the subgroup lattice of D_8 and marking via horizontal lines those subgroups that are G-conjugate for different choices of Sylow supergroups G:



where $P_1 \cong P_2 \cong C_2^2$, $P_3 \cong C_4$, $Q_i \cong C_2$ for $1 \leq i \leq 5$, and $Q_5 = Z(D_8)$. In this picture, the decoration above a horizontal line indicates the smallest fusion system in which particular subgroups become fused.

Of course, the fusion system is more than just the data of which subgroups are fused; the morphisms that do the fusing must also be recalled. In all of the fusion systems, $\mathrm{Aut}_{\mathcal{F}}(D_8) = \mathrm{Aut}_{D_8}(D_8) \cong C_2^2$ and $\mathrm{Aut}_{\mathcal{F}}(P_3) = \mathrm{Aut}_{D_8}(P_3) \cong C_2$. For $i = 1, 2$, if not all the order 2 subgroups of P_i are fused in $\mathcal{F}$, then $\mathrm{Aut}_{\mathcal{F}}(P_i) = \mathrm{Aut}_{D_8}(P_i) \cong C_2$; else $\mathrm{Aut}_{\mathcal{F}}(P_i) \cong \Sigma_3$.

Let $\alpha_1 \in \mathrm{Aut}(P_1)$ be the automorphism that sends Q_1 to Q_5, and let $\alpha_2 \in \mathrm{Aut}(P_2)$ send Q_5 to Q_4. For $1 \leq i, j \leq 5$, let $\beta_{ij} : Q_i \to Q_j$ be the unique isomorphism.

Let's examine how far isomorphisms of these fusion systems can be extended:

- Every morphism in $\mathcal{F}_{D_8}(D_8)$ extends to an automorphism of D_8, by construction.
- $\beta_{15} = \alpha_1|_{Q_1}$, but α_1 cannot extend to an automorphism of D_8 (it does not fix the center, which is characteristic), but at least every morphism in $\mathcal{F}_{D_8}(\Sigma_4)$ is the restriction of the automorphism of some object.
- In $\mathcal{F}_{D_8}(A_6)$, the morphism β_{14} is not the restriction of *any* automorphism of a supergroup containing Q_1 and Q_4 (such a supergroup would have to be D_8, so the automorphism would have to be an outer automorphism of D_8, but $\mathrm{Out}(D_8) \cong C_2$, thus if there were such an extension it would violate the fact that $D_8 \in \mathrm{Syl}_2(A_6)$.) However, we can write β_{14} as the composite of restrictions of automorphisms: $\beta_{14} = \alpha_2|_{Q_5} \circ \alpha_1|_{Q_1}$.

It turns out this last situation is as bad as it gets for general $\mathcal{F}_S(G)$.

Theorem 3 (Alperin's fusion theorem)**.** *For $S \in \mathrm{Syl}_p(G)$, let $\varphi : P \to Q$ be an isomorphism in $\mathcal{F}_S(G)$. Then there are subgroups $R_1, R_2, \ldots, R_n \leq S$ and automorphism $\alpha_i \in \mathrm{Aut}(R_i)$ such that $P \leq R_1$, $Q \leq R_n$, and*

$$\varphi = \alpha_n|_{\alpha_{n-1}\ldots\alpha_2\alpha_1(P)} \circ \ldots \circ \alpha_2|_{\alpha_1(P)} \circ \alpha_1|_P.$$

This is the key local-to-global structure property of fusion systems: Knowledge of the "local data" of the automorphisms of objects (which are quotients $N_G(R)/C_G(R)$ for R a p-group) implies complete knowledge of the "global data" of G-conjugacy, a.k.a. all morphisms of the fusion system.

Abstract fusion systems

We will pause before proving this result, so that we can take a slightly more general perspective and ultimately prove something stronger. One of the key insights of Puig was that, when we're working with fusion systems, we don't really need to include G in the discussion. This leads to the following notion:

Definition 4. An *abstract fusion system* on the p-group S is a category $\mathcal{F}$ with objects the subgroups of S, and whose morphisms satisfy $\mathrm{Hom}_S(P, Q) \subseteq \mathcal{F}(P, Q) \subseteq \mathrm{Inj}(P, Q)$. Moreover, we require that if $\varphi \in \mathcal{F}(P, Q)$ is an isomorphism of groups, then $\varphi^{-1} \in \mathcal{F}(Q, P)$.

Note that the last requirement implies that every morphism for $\mathcal{F}$ factors uniquely as an isomorphism followed by an inclusion.

Abstract fusion systems model the inclusion of a p-group S in some ambient supergroup G (in fact, a recent result of Park proves that every abstract fusion system arises in this manner), but it does not reflect the added structure of the case that G is a *Sylow* supergroup. (In our D_8 examples, it is never possible that β_{14} extend to an automorphism of D_8 induced by conjugation in a Sylow supergroup, for the reason already mentioned.) For that we need a little terminology...

Definition 5. Let $\mathcal{F}$ be an abstract fusion system on S and $P, Q \leq S$.

- P is *fully $\mathcal{F}$-centralized* if $|C_S(P)| \geq |C_S(\varphi P)|$ for all $\varphi \in \mathcal{F}(P, S)$.
  [If $S \in \mathrm{Syl}_p(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$, this is equivalent to $C_S(P) \in \mathrm{Syl}_p(C_G(P))$.]
- P is *fully $\mathcal{F}$-normalized* if $|N_S(P)| \geq |N_S(\varphi P)|$ for all $\varphi \in \mathcal{F}(P, S)$.
  [If $S \in \mathrm{Syl}_p(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$, this is equivalent to $N_S(P) \in \mathrm{Syl}_p(N_G(P))$.]
- If $\varphi \in \mathcal{F}(P, Q)$ is an isomorphism, the *extender* of φ the subgroup of $N_S(P)$ given by $N_\varphi := \{n \in N_S(P) | \varphi \circ c_n \circ \varphi^{-1} \in \mathrm{Aut}_S(Q)\}$.
  [N_φ is the maximal subgroup of $N_S(P)$ to which φ could possibly extend.]
  [Note that $P \cdot C_S(P) \leq N_\varphi$ for all φ.]

... and some additional axioms:

Definition 6. The abstract fusion system $\mathcal{F}$ on S is *saturated* if

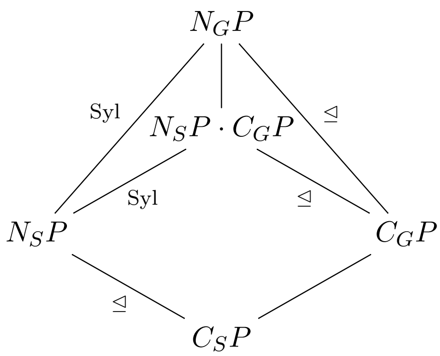
(Syl) If P is fully $\mathcal{F}$-normalized, then $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.
(N⇒C) If P is fully $\mathcal{F}$-normalized, then P is fully $\mathcal{F}$-centralized.
(Ext) If $\varphi \in \mathcal{F}(P, Q)$ is an isomorphism and Q is fully $\mathcal{F}$-centralized, then φ extends to some $\widetilde{\varphi} \in \mathcal{F}(N_\varphi, S)$.

Proposition 7. *If $S \in \mathrm{Syl}_p(G)$, then $\mathcal{F}_S(G)$ is saturated.*

Proof. Set $\mathcal{F} := \mathcal{F}_S(G)$.

Axiom (Syl) follows from the fact that P is fully $\mathcal{F}$-normalized iff $N_S(P) \in \mathrm{Syl}_p(N_G(P))$.

To verify Axiom (N⇒C), suppose that P is fully $\mathcal{F}$-normalized, and consider the following diagram of subgroups of G:



$C_G P \trianglelefteq N_G P$, so $N_S P \cdot C_G P$ is a subgroup of $N_G P$. The inclusion $N_S P \leq N_G P$ is Sylow, so the inclusion $N_S P \leq N_S P \cdot C_G P$ is as well. $C_S P = N_S P \cap C_G P$, so the lattice isomorphism theorem implies $N_S P \cdot C_G P / C_G P \cong N_S P \cap C_S P$, and in particular both indices are equal. Therefore the inclusion $N_S P \leq N_S P \cdot C_G P$ is Sylow iff the inclusion $C_S P \leq C_G P$ is, and we conclude that P is fully $\mathcal{F}$-centralized.

To verify Axiom (Ext), fix an isomorphism $\varphi \in \mathcal{F}(P, Q)$ with Q fully $\mathcal{F}$-centralized. Fix $g \in G$ with $\varphi = c_g|_P$. It follows from the definition of N_φ that ${}^g N_\varphi \leq N_S Q \cdot C_G Q$. The same diagram as above (with P replaced by Q) shows that $C_S Q \leq C_G Q$ is Sylow iff $N_S Q \leq N_S Q \cdot C_G Q$ is, so there is some $h \in C_G Q$ such that ${}^{hg} N_\varphi \leq N_S Q$. It follows immediately that φ extends to $c_{hg} \in \mathcal{F}(N_\varphi, S)$. □

Note that, while every Sylow inclusion gives rise to a saturated fusion system, not every saturated fusion system can be realized as a Sylow inclusion into a finite group. Saturated fusion systems that do not arise from finite groups in this way are called *exotic*. The most famous example was discovered by Solomon, who should that the Sylow 2-subgroup on $\mathrm{Spin}_3(7)$ (of order 2^{10}) has a saturated fusion system in which all involutions are conjugate, but that this cannot be realized by any finite group. More recently, using the Classification of Finite Simple Groups, Ruiz and Viruel showed that on the extra special group 7^{1+2}_+ there are exotic fusion systems as well, and since then there has been a good number of exotic fusion systems discovered at odd primes.

An important consequence of the saturation axioms is the following:

Proposition 8. *Suppose that $P, Q \leq S$ are isomorphic in $\mathcal{F}$.*

- *If Q is fully $\mathcal{F}$-centralized, there is a morphism $C_S P \to C_S Q$ in $\mathcal{F}$ that restricts to an isomorphism in $\mathcal{F}(ZP, ZQ)$.*
- *If Q is fully $\mathcal{F}$-normalized, there is a morphism $N_S P \to N_S Q$ in $\mathcal{F}$ that restricts to an isomorphism in $\mathcal{F}(P, Q)$.*

Proof. Let $\varphi \in \mathcal{F}(P, Q)$ be an isomorphism.

- $C_S P \leq N_\varphi$, so the first assertion follows from Axiom (Ext).
- Set $\overline{N} := \{\varphi \circ c_n \circ \varphi^{-1} | n \in N_S P\} \leq \mathrm{Aut}_\mathcal{F}(Q)$. Q is fully $\mathcal{F}$-normalized, so by Axiom (Syl), $\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_\mathcal{F}(Q))$. Thus there is some $\chi \in \mathrm{Aut}_\mathcal{F}(Q)$ such that ${}^\chi\overline{N} \leq \mathrm{Aut}_S(Q)$. It follows immediately that $N_S P = N_{\chi\varphi}$ and $\chi\varphi$ extends to some morphism in $\mathcal{F}(N_S P, N_S Q)$ that sends P to Q.

□

Thus in saturated fusion systems, saying that a subgroup is fully $\mathcal{F}$-centralized (or -normalized) means not just that the size of the centralizer (or normalizer) is maximal in an isomorphism class, but that there is an actual injection of centralizers (or normalizers) that realizes all other size bounds.

Corollary 9. *If P, Q are fully $\mathcal{F}$-centralized and isomorphic in $\mathcal{F}$, then $C_S P$ and $C_S Q$ are isomorphic in $\mathcal{F}$ as well. The analogous result holds when P and Q are fully $\mathcal{F}$-normalized.*

Proving the Fusion Theorem

We now set out to prove the strong version of Alperin's Fusion Theorem, which not only proves that all morphisms are composites of restricted automorphisms, but identifies the groups that give rise to the generating automorphisms.

Definition 10. For $P \lneq S$, let $\mathrm{Aut}^+_\mathcal{F}(P) \leq \mathrm{Aut}_\mathcal{F}(P)$ be the subgroup of automorphisms that can be written as restrictions of automorphisms of strictly larger subgroups. P is *$\mathcal{F}$-essential* if $\mathrm{Aut}^+_\mathcal{F}(P) \neq \mathrm{Aut}_\mathcal{F}(P)$.

Theorem 11. *Let $\mathcal{C}$ be the set of fully $\mathcal{F}$-normalized, $\mathcal{F}$-essential subgroups of S, together with S itself. Then every isomorphism of $\mathcal{F}$ can be written as a composite of restrictions of automorphisms of subgroups in $\mathcal{C}$.*

Proof. Suppose not, and let $\varphi \in \mathcal{F}(P, Q)$ be an isomorphism that cannot be written in terms of automorphisms of subgroups in $\mathcal{C}$ such that $|P|$ is maximal subject to being the source of such an isomorphism.

First I claim that we may assume that Q is fully $\mathcal{F}$-normalized; if not, pick an isomorphism $\psi \in \mathcal{F}(Q, R)$ with R fully $\mathcal{F}$-normalized, and consider

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ & \overset{\psi\varphi}{\searrow} & \downarrow \psi \\ & & R \end{array}$$

Clearly if ψ can be written as a composition from $\mathcal{C}$, then ψ^{-1} can as well. Similarly, if $\psi\varphi$ can be written as a composition from $\mathcal{C}$, then $\psi^{-1} \circ \psi\varphi = \varphi$ can, so our first reduction holds.

Next I claim that we may assume $P = Q$. If not, there is some $\psi \in \mathcal{F}(N_S P, N_S Q)$ because Q is fully $\mathcal{F}$-normalized. Since $P \lneq S$, we have $P \lneq N_S P$, so the maximality of φ implies that ψ can be written as restrictions from $\mathcal{C}$. If $\varphi\psi|_P^{-1} \in \mathrm{Aut}_{\mathcal{F}}(Q)$ can be written as restrictions from $\mathcal{C}$, then $\varphi = (\varphi\psi|_P^{-1}) \circ \psi|_P$ can as well. But if $\varphi\psi|_P^{-1}$ cannot be written as a restriction of automorphisms of larger subgroups, this means that Q is $\mathcal{F}$-essential, and we're done. □

This is technically stronger than the original iteration of the Fusion Theorem, but it's a little too circular: Yes, it's true that every isomorphism (and hence every morphism) of $\mathcal{F}$ is the composite of restricted automorphisms, but it would be nice to have a better definition of $\mathcal{F}$-essential subgroups (beyond "those that have an automorphism that isn't the composite of automorphisms of larger subgroups). Luckily, we can find a somewhat more intrinsic formulation. We begin by observing a few properties of $\mathcal{F}$-essential subgroups.

Proposition 12. *If P is fully $\mathcal{F}$-normalized and $\mathcal{F}$-essential, then $C_S P \leq P$.*

Proof. By Axiom (Ext), every $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ has an extension $\widetilde{\varphi} \in \mathcal{F}(N_\alpha, S)$. It is clear that $C_S P \leq N_\alpha$, so if $\mathrm{Aut}_{\mathcal{F}}^{+}(P) \neq \mathrm{Aut}_{\mathcal{F}}(P)$ we must have $C_S P \leq N_\alpha = P$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}}^{+}(P) \setminus \mathrm{Aut}_{\mathcal{F}}(P)$. □

There is a term for this:

Definition 13. A subgroup $P \leq S$ is *$\mathcal{F}$-centric* if $C_S P' \leq P'$ for all $P' \cong_{\mathcal{F}} P$.

Note that if P is fully $\mathcal{F}$-normalized, then P is fully $\mathcal{F}$-centralized, and our earlier comment about all centralizers mapping into the centralizer of P implies that $C_S P \leq P$ is sufficient for P to be $\mathcal{F}$-centric.

Proposition 14. *Suppose that P is fully $\mathcal{F}$-normalized and $\mathcal{F}$-essential. Then for every $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \setminus \mathrm{Aut}_{\mathcal{F}}^{+}(P)$, we have $\mathrm{Inn}(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}^{+}(P) \cap {}^{\alpha}\mathrm{Aut}_{\mathcal{F}}^{+}(P))$.*

Proof. If $\Gamma \gneq \mathrm{Inn}(P)$ is Sylow in $\mathrm{Aut}_{\mathcal{F}}^{+}(P) \cap {}^{\alpha}\mathrm{Aut}_{\mathcal{F}}^{+}(P)$, we have $\Gamma, {}^{\alpha^{-1}}\Gamma \in \mathrm{Aut}_{\mathcal{F}}^{+}(P)$. Since $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}^{+}(P))$, there are $\beta, \delta \in \mathrm{Aut}_{\mathcal{F}}^{+}(P)$ such that ${}^{\beta}\Gamma, {}^{\delta\alpha^{-1}}\Gamma \leq \mathrm{Aut}_S(P)$. Let R be the preimage of ${}^{\beta}\Gamma$ in $N_S P$, so $P \lneq R \leq N_S P$, and $R \leq N_{\delta\alpha^{-1}\beta^{-1}}$. Thus $\delta\alpha^{-1}\beta^{-1}$ extends to R, so $\delta\alpha^{-1}\beta^{-1} \in \mathrm{Aut}_{\mathcal{F}}^{+}(P)$. It follows that $\alpha \in \mathrm{Aut}_{\mathcal{F}}^{+}(P)$, contrary to our assumption. □

There is also a technical term that describes this, but is most easily stated after modding out by $\mathrm{Inn}(P) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(P)$. We denote the resulting quotient by $\mathrm{Out}_{\mathcal{F}}(P)$.

Definition 15. Let G be a finite group and $H \lneq G$. Then H is a *strongly p-embedded subgroup* if H contains a Sylow p-subgroup of G but $p \nmid |H \cap {}^{g}H|$ for all $g \in G \setminus H$.

Thus the previous proposition says that $\mathrm{Out}_{\mathcal{F}}(P)$ has a strongly p-embedded subgroup when P is $\mathcal{F}$-essential, namely the image of $\mathrm{Aut}^{+}_{\mathcal{F}}(P)$.

We can relate this to yet another technical term, whose use will show up more later.

Definition 16. For G a finite group, denote by $O_p(G)$ the maximal normal p-subgroup of G.

For $P \leq S$ and $\mathcal{F}$ a fusion system on S, we say P is *$\mathcal{F}$-radical* if $O_p(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Inn}(P)$, or equivalently if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$. In this latter case we say that $\mathrm{Out}_{\mathcal{F}}(P)$ is *p-reduced.*

Lemma 17. *G contains a strongly p-embedded subgroup H if and only if the poset of nonidentity p-subgroups of G is disconnected.*

Proof. If H is strongly p-embedded in G, I claim that for all p-subgroups $P \leq G$ such that $P \cap H \neq 1$ we must have $P \leq H$. Pick $S, T \in \mathrm{Syl}_p(G)$ such that $P \cap H \leq S \leq H$ and $P \leq T$, and let g conjugate S to T. Then $1 \neq P \cap H \leq H \cap {}^{g}H$, so it follows that $g \in H$ and $P \leq H$. An easy induction then implies that every p-subgroup of G in the same component as a p-subgroup of H must in fact be contained in H as well. Thus the p-subgroups of H and ${}^{g}H$ must live in different components for $g \notin H$.

Conversely, suppose that the lattice is disconnected, and let H be the G-stabilizer of one of the components. Clearly H is proper in G, contains a Sylow subgroup, and has p' intersection with any of its distinct conjugates. □

Corollary 18. *If P is $\mathcal{F}$-essential, then P is $\mathcal{F}$-radical.*

Proof. If P is $\mathcal{F}$-essential, we've seen that $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup, which means that the nonidentity p-subgroup lattice of $\mathrm{Out}_{\mathcal{F}}(P)$ has at least two components. But if $1 \neq K \trianglelefteq \mathrm{Out}_{\mathcal{F}}(P)$ for a p-group, then K would be contained in every component. Thus we must have $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$, as desired. □

This leads to the Alperin-Goldschmidt version of the Fusion Theorem:

Theorem 19. *If $\mathcal{F}$ is a saturated fusion system on S, then the automorphisms of the $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroups determine $\mathcal{F}$.*

In fact, the result that P a fully $\mathcal{F}$-normalized, $\mathcal{F}$-essential subgroup can be inverted. First we need:

Lemma 20. *Suppose that G is a finite group with $S \in \mathrm{Syl}_p(G)$. Set*

$$H := \langle N_G P | 1 \neq P \leq S \rangle.$$

If $H \lneq G$, then H is a strongly p-embedded subgroup, and H is minimal with respect to those p-embedded subgroups that contain H.

Proof. Suppose that $T \in \mathrm{Syl}_p(H \cap {}^{g}H)$. Then, as $S \in \mathrm{Syl}_p(H)$, there is some $a \in H$ such that $P := {}^{a}T \leq S$. Similarly, since ${}^{g^{-1}}T \leq H$, there is some $b \in H$ such that $Q := {}^{bg^{-1}}T \leq S$. Thus P is a subgroup of S that is conjugated into S by $bg^{-1}a^{-1} \in G$. The weak form of Alperin's fusion theorem shows that every element of G that conjugates a nontrivial subgroup of S into S can be written as a product of elements in the normalizers of nontrivial subgroups of S, so $bg^{-1}a^{-1} \in H$. As $a, b \in H$, it follows that $g \in H$ as well, a contradiction unless T is trivial. Thus H is strongly p-embedded in G.

Now, suppose that L is some other strongly p-embedded subgroup of G that contains S. If there is some $1 \neq P \leq S$ such that $N_G P \not\leq L$, then for any $n \in N_G P \backslash L$, we have $1 \neq P \leq L \cap {}^{n}L$, so L cannot be strongly p-embedded. □

Finally we come to the strongest version of the Fusion Theorem, due to Puig, which characterizes the $\mathcal{F}$-essential subgroups in terms of the fusion system.

Theorem 21. *Let $P \leq S$ be fully $\mathcal{F}$-normalized. Then P is $\mathcal{F}$-essential if and only if P is $\mathcal{F}$-centric and $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup.*

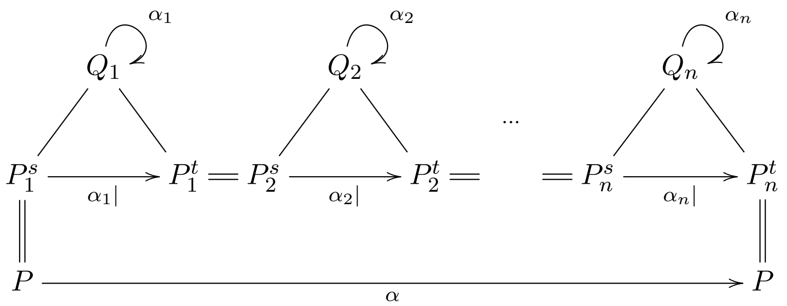
Proof. We have already seen that if P is fully $\mathcal{F}$-normalized and $\mathcal{F}$-essential, then P is $\mathcal{F}$-centric and $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup. We show now the opposite implication.

Suppose that $G := \mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup H, without loss of generality taken to contain $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(G)$. I claim that the image of $\mathrm{Aut}^+_{\mathcal{F}}(P)$ in G is contained in the minimal strongly embedded p-subgroup containing $\mathrm{Out}_S(P)$. Suppose that $\alpha \in \mathrm{Aut}^+_{\mathcal{F}}(P)$, so that

$$\alpha = \alpha_n|_{\alpha_{n-1}\ldots\alpha_2\alpha_1(P)} \circ \ldots \circ \alpha_2|_{\alpha_1(P)} \circ \alpha_1|_P$$

where $\alpha_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$ for subgroups Q_i strictly larger that P.

Set $P_1^s = P$, $P_1^t = \alpha_1(P) = P_2^s$, $P_2^t = \alpha_2\alpha_1(P) = P_3^s$, etc., so we have the following commutative diagram:

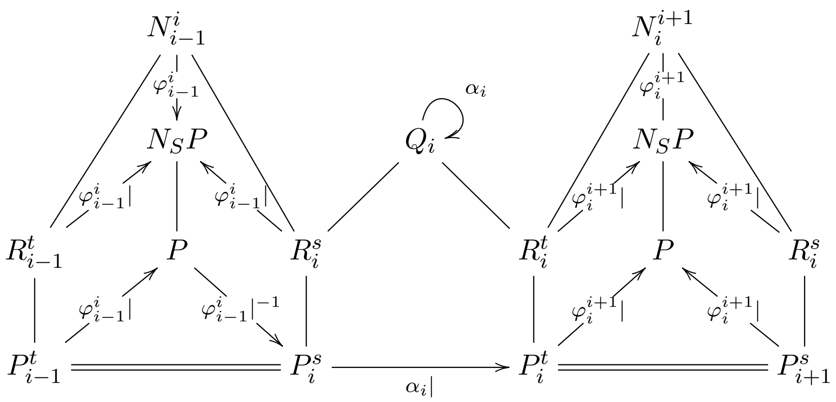


We are going to write α as a composite of automorphisms of P by further resolving the equalities of this diagram.

For each $1 \leq i \leq n-1$, let N_i^{i+1} be the normalizer in S of $P_i^t = P_{i+1}^s$. As P is fully $\mathcal{F}$-normalized, there exist morphisms in $\varphi_i^{i+1} \in \mathcal{F}(N_i^{i+1}, N_S)$ that send $P_i^t = P_{i+1}^s$ to P; fix one such morphism for each i. Set

$$R_i^s = N_{i-1}^i \cap Q_i = N_{Q_i}(P_i^s) \text{ and } R_i^t = N_i^{i+1} \cap Q_i = N_{Q_i}(P_i^t).$$

Then we can resolve the original factorization of α further; at stage i, the new diagram looks as follows:



Thus we can further decompose α as

$$\alpha = \alpha_n \varphi_{n-1}^n|^{-1} \circ \varphi_{n-1}^n \alpha_{n-1} \varphi_{n-2}^{n-1}|^{-1} \circ \ldots \circ \varphi_2^3 \alpha_2 \varphi_1^2|^{-1} \circ \varphi_1^2 \alpha_1,$$

where we have omitted most restriction signs in an attempt readability. (We deal with the edge cases by setting $Q_0 = Q_{n+1} = S$, $\alpha_0, \alpha_{n+1} = \mathrm{id}_S$, and $\varphi_0^1 = \varphi_n^{n+1} = \mathrm{id}_{N_S P}$.) Each term $\psi_i := \varphi_i^{i+1} \alpha_i \varphi_{i-1}^i|^{-1}$ lies in $\mathrm{Aut}_{\mathcal{F}}(P)$; we will show that the image $\overline{\psi}_i \in \mathrm{Out}_{\mathcal{F}}(P)$ of ψ_i is the product of elements that normalize nontrivial subgroups of $\mathrm{Out}_S(P) \in \mathrm{Syl}_p(\mathrm{Out}_{\mathcal{F}}(P))$. As P is properly contained in both $\varphi_i^{i+1}(R_i^t)$ and $\varphi_{i-1}^i(R_i^s)$, the image of both of these groups in $\mathrm{Out}_S(P)$ is nontrivial. Since $\alpha_i(R_i^s) = R_i^t$, ψ_i conjugates the image of $\varphi_{i-1}^i(R_i^s)$ to the image of $\varphi_i^{i+1}(R_i^t)$, so these p-groups are fused in $\mathcal{F}_{\mathrm{Out}_S(P)}(\mathrm{Out}_{\mathcal{F}}(P))$ via $\overline{\psi}_i$. By the Fusion Theorem, we can write $\overline{\psi}_i$ as the product of elements that normalize $\mathcal{F}_{\mathrm{Out}_S(P)}(\mathrm{Out}_{\mathcal{F}}(P))$-centric, and hence nontrivial, subgroups.

Thus $\mathrm{Out}_{\mathcal{F}}^+(P)$ is contained in the minimal strongly p-embedded subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$ that contains $\mathrm{Out}_S(P)$. By assumption that $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup, it follows that $\mathrm{Out}_{\mathcal{F}}^+(P) \neq \mathrm{Out}_{\mathcal{F}}(P)$, as thus $\mathrm{Aut}_{\mathcal{F}}^+(P) \neq \mathrm{Aut}_{\mathcal{F}}(P)$, so that P is $\mathcal{F}$-essential.

□