Discrete Mathematics lecture notes 7-2

October 17, 2013

23. Trinomial coefficients

Last time we saw that the binomial coefficients $\binom{n}{i}$ we had introduced earlier could be used to find a formula for powers of sums of two numbers:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

While the Binomial Theorem is certainly a nice result, if you look at it the right way it's basically a proof by definition: We might guess that there should exist *some* numbers $c_{n,i}$ such that

$$(a+b)^n = \sum_{i=0}^{n} c_{n,i} \cdot a^i b^{n-i}$$

would hold for all i, n; our choice $c_{n,i} = \binom{n}{i}$, which was really the choice of the definition of the symbol $\binom{n}{i}$, was really motivated by wanting the Binomial Theorem to be true.

In a similar manner, we could ask for symbols $\binom{n}{i,j,k}$ that would yield a Trinomial Theorem:

$$(a+b+c)^n = \sum_{\substack{i,j,k\in\mathbb{N} \\ i+j+k=n}} \binom{n}{i,j,k} a^i b^j c^k$$

for all $n \in \mathbb{N}$.

Definition 1. For $n \in \mathbb{N}$ and $i, j, k \in \mathbb{N}$ such that $i + j + k = n$, define the *trinomial coefficient* $\binom{n}{i,j,k}$ to be the number of ordered triples of subsets of $[n]$,[1] say (X, Y, Z), such that $X \cup Y \cup Z = [n]$ and $|X| = i$, $|Y| = j$, and $|Z| = k$.

Exercise. Show that if $X, Y, Z \subseteq [n]$ are such that $X \cup Y \cup Z = [n]$ and $|X| + |Y| + |Z| = n$, then X, Y, and Z are pairwise disjoint.

It is a big chunk of your next homework set to explore the properties of these trinomial coefficients. We will just note here that we can form a higher dimensional analogue of Pascal's Triangle using trinomial coefficients: The resulting *Pascal's Pyramid* has at its nth level all trinomial coefficients of the form $\binom{n}{i,j,k}$ where i, j, and k are free to vary subject only that they remain in $\mathbb{N}$ and sum to n. So the first layer has just $\binom{0}{0,0,0}$, the next has $\binom{1}{1,0,0}$, $\binom{1}{0,1,0}$, and $\binom{1}{0,0,1}$, etc.

One can imagine taking the notion of trinomial coefficients another step higher, obtaining symbols of the form $\binom{n}{i,j,k,\ell}$, and then another step, etc. Exploring the properties of these "higher dimensional" analogues of Pascal's Triangle would be a good exercise.

23. Partitions and equivalence relations

Let's fix a set X and a partition $\mathcal{A} := \{A_1, A_2, \ldots, A_n\}$. Recall that being a *partition* means that

- $A_i \neq \emptyset$ for all i,
- $A_i \cap A_j = \emptyset$ if $i \neq j$, and
- $\bigcup_{i=1}^{n} A_i = X$.

[1] Recall that $[n] := \{1, 2, 3, \ldots, n\}$.

Using this partition, we define a particular relation $\sim_{\mathcal{A}}$ on X by:

$$x \sim_{\mathcal{A}} y \text{ iff there is some } i \text{ such that } x \text{ and } y \in A_i.$$

This gives a well-defined relation, which happens to have many nice properties:

Proposition 2. *$\sim_{\mathcal{A}}$ is reflexive, symmetric, and transitive.*

Proof. If $x \in X$, then there is some $j \in [n]$ such that $x \in A_j$ by the assumption that the union of the A_i is X. But then $x \in A_j$ and $X \in A_j$, so $x \sim_{\mathcal{A}} x$ and $\sim_{\mathcal{A}}$ is reflexive.

If $x \sim_{\mathcal{A}} y$, then x and y lie in the same A_i for some i, meaning y and x lie in the same A_i and $y \sim_{\mathcal{A}} x$. Thus $\sim_{\mathcal{A}}$ is symmetric.

If $x \sim_{\mathcal{A}} y$, then x and y live in A_i for some i. If $y \sim_{\mathcal{A}} z$ then y and z live in A_j for some j. In particular, $y \in A_i \cap A_j$, so by the assumption that the subsets of the partition are pairwise disjoint, we must have $i = j$. But this means x and z live in the same $A_i = A_j$, or $z \sim_{\mathcal{A}} z$. ☐

Such a relation has a special name:

Definition 3. An *equivalence relation* on a set X is a relation $\sim$ on X that is reflexive, symmetric, and transitive. If $x \sim y$, we say "x is equivalent to y."

We've just seen that any partition gives rise to an equivalence relation; in fact, the converse also holds:

Definition 4. Let $\sim$ be an equivalence relation on X, and for each $x \in X$, the *equivalence class* of x is $[x] := \{y \in X | x \sim y\}$.

Proposition 5. *If x and y are elements of X, then either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.*

Proof. Suppose that $[x] \cap [y] \neq \emptyset$, so there is some $z \in [x] \cap [y]$. Then $x \sim z$ and $y \sim z$. As $\sim$ is symmetric, we must have $z \sim y$ as well; as $\sim$ is transitive, it follows that $x \sim y$. Therefore it suffice to show that if $x \sim y$, we have $[x] = [y]$. Prove this as an exercise. ☐

Proposition 6. *Let $\sim$ be an equivalence relation on X, and let $X/\sim$ denote the set of distinct equivalence classes of $\sim$. Then the elements of $X/\sim$[2] form a partition of X.*

Proof. Clearly none of the equivalence classes is empty, as $x \in [x]$ (using reflexivity of $\sim$). Distinct equivalence classes are disjoint by the previous Proposition (which you'll note used both symmetry and transitivity of $\sim$). And finally, if $x \in X$, again we have $x \in [x]$, so that X is the union of the classes. ☐

Thus the notion of equivalence relation and partition contain exactly the same information, and we can freely move back and forth between them as necessary. Let's close with an example that we have already dealt with:

Example 7. Pick $n \in \mathbb{N}$, and let $\sim_n$ be the relation on $\mathbb{Z}$ given by $a \sim_n b$ if $n | b - a$. Show that $\sim_n$ is an equivalence relation. If $n \geq 1$, there are n equivalence classes, namely

$$\{0, \pm n, \pm 2n, \dots\}, \{1, 1 \pm n, 2 \pm 2n, \dots\}, \{2, 2 \pm n, 2 \pm 2n, \dots\}, \dots, \{n-1, n-1 \pm n, n-1 \pm 2n, \dots\}.$$

In other words, what we had described earlier as $\mathbb{Z}/n$ was, as a set, secretly always the equivalence classes of the "divisible by n" relation.

Exercise. How many equivalence classes are there for $\sim_0$? Can you describe $\sim_0$ explicitly?

[2] Recall that these are subsets of X, by definition