Discrete Mathematics lecture notes 6-1

October 10, 2013

19. Counting with infinite sets

We've now see that we can define the cardinality of a finite set in terms of bijections with certain special sets of finite size, the $\{[n]\}_{n\in\mathbb{N}}$. In other words, we can think of the natural number as finite sets minus all the data other than the size (so we don't remember that the sets $A = \{a, b, c\}$ and $[3] = \{1, 2, 3\}$ are not equal, even though they have the same size). Going the other ways, from numbers to sets, is a process that is known as *categorification*; this is an active area of research, at least in more complicated settings.

But let's leave all that aside for the moment. For now, what, if anything, does this mean in the case of *infinite* sets?

Definition 1. A set A is *infinite* if there is no $n \in \mathbb{N}$ and a bijection $\varphi : [n] \xrightarrow{\sim} A$. In other words, A is infinite if A is not finite.

By definition, we can't count all the elements of an infinite set. We can, however, say when two infinite sets are the same size, thanks to rigor with which we introduced finite cardinals:

Definition 2. Two sets A and B are *the same cardinal* (or *size*, or...) if there exists a bijection $\varphi : A \xrightarrow{\sim} B$. In this case we will write $|A| = |B|$.

Remark 3. We might also define notions like $\leq$ or $\geq$ for cardinals, by saying $|A| \leq |B|$ if there is some injection $\psi : A \hookrightarrow B$ and $|A| \geq |B|$ if there is some surjection $\chi : A \twoheadrightarrow B$. Of course, after making these definitions we should prove a bunch of theorems of the form "If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$." This is possible,[1] but we will not have need of this level of generality.

Example 4 ($|\mathbb{N}| = |\mathbb{N}_{>0}|$). Set $\mathbb{N}_{>0} := \{n \in \mathbb{N} | n > 0\} \subsetneq \mathbb{N}$.[2] Consider the maps

$$f : \mathbb{N} \to \mathbb{N}_{>0} : n \mapsto n + 1 \qquad \text{and} \qquad g : \mathbb{N}_{>0} \to \mathbb{N} : m \mapsto m - 1.$$

It is easy to check that both of these maps are well-defined,[3] and that we have $g \circ f = \mathrm{id}_{\mathbb{N}}$ and $f \circ g = \mathrm{id}_{\mathbb{N}_{>0}}$. This means that that $g = f^{-1}$, and[4] therefore f and g are both bijections. We conclude that $|\mathbb{N}| = |\mathbb{N}_{>0}|$.

This example shows that it is possible for an infinite set to be the same size as a proper subset, a statement that would be impossible for finite set by the Pigeonhole Principle.

Example 5 ($|\mathbb{N}| = |\mathbb{Z}|$). Consider the map

$$f : \mathbb{N} \to \mathbb{Z} : n \mapsto \begin{cases} -k & n = 2k,\ k \in \mathbb{N} \\ k + 1 & n = 2k + 1,\ k \in \mathbb{N} \end{cases} .$$

Then f can be represented

n	0	1	2	3	4	5	6	...
$f(n)$	0	1	−1	2	−2	3	−3	...

and it is not hard to see that f is bijective. Thus $|\mathbb{N}| = |\mathbb{Z}|$.

Exercise. Show that the map $f : \mathbb{N} \to \mathbb{Z}$ of the previous exercise is a bijection by finding an explicit inverse function $g : \mathbb{Z} \to \mathbb{N}$.

Example 6 ($|\mathbb{N}| = |\mathbb{Q}|$). We will actually show that $|\mathbb{N}| = |\mathbb{Q}_{\geq 0}|$, i.e., the sets of natural numbers and nonnegative rational numbers have the same cardinality. We begin by listing all positive rational numbers

[1] And a good exercise, though a bit tough.
[2] The symbol $\subsetneq$ indicates a *proper* subset, i.e., a subset which is the whole of the original set.
[3] I.e., they make sense, in the context of the indicated domains and ranges.
[4] Cf. our earlier investigation of the relationship between inverses and bijections,

in a doubly infinite grid whose (i,j)-entry is $\frac{i}{j}$:

0	1	2	3	4	5	$\cdots$
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\cdots$
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\cdots$
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\cdots$
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\cdots$
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

or, without repeats

0	1	2	3	4	5	$\cdots$
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\cdots$
2	$\frac{2}{1}$		$\frac{2}{3}$		$\frac{2}{5}$	$\cdots$
3	$\frac{3}{1}$	$\frac{3}{2}$		$\frac{3}{4}$	$\frac{3}{5}$	$\cdots$
4	$\frac{4}{1}$		$\frac{4}{3}$		$\frac{4}{5}$	$\cdots$
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$		$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Next, we define our function $f : \mathbb{N} \to \mathbb{Q}_{\geq 0}$ by first setting $f(0) = 0$, and then counting off all the rational numbers by traveling along the diagonals of slope 1.[5] Thus we get:

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$f(n)$	0	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{1}{2}$	$\frac{3}{1}$	$\frac{1}{3}$	$\frac{4}{1}$	$\frac{3}{2}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{5}{1}$	$\cdots$

Because every positive rational number appears in our grid, we have f is surjective, and because we deleted all the repeat labelings of the same rational number, f is injective.

Exercise. Show that $|\mathbb{N}| = |\mathbb{Q}_{\geq 0}|$ implies $|\mathbb{N}| = |\mathbb{Q}|$.

Ok, at this point one could conclude that the notion of infinite sets' having the same cardinality is not really that useful a definition: After all, if two sets are infinite, it seems[6] that you should always be able to find a bijection between them. One might feel justified in making this leap, but one would be wrong.

Example 7 ($|\mathbb{N}| \neq |\mathbb{R}|$). Clearly, as $\mathbb{N} \subseteq$, we must have $|\mathbb{N}| \leq |\mathbb{R}|$, so what we're really saying is

Proposition 8. *There exist no surjections $f : \mathbb{N} \twoheadrightarrow \mathbb{R}$.*

Proof. We will actually show that there are no surjections $f : \mathbb{N}_{>0} \twoheadrightarrow (0,1)$.[7] This is rather more difficult than the earlier examples; before, when showing that we did have two sets of equal cardinality, we just[8] needed to find some bijection; here, we need to show that no surjection can exist. Proof by contradiction seems to be the natural tool to use, so let's give it a shot.

Suppose that we have a surjection $f : \mathbb{N}_{>0} \twoheadrightarrow (0,1)$. Let's list the real numbers that f is counting off in terms of their decimal expansion. In order to avoid technical difficulties,[9] we require that any decimal expansion *not* end in an infinite string of 9s. In other words, given the choice between representing $\frac{1}{4}$ as $0.25000000\ldots$ and $0.24999999\ldots$, go with the former.

So, f gives us a list of decimals:

n	$f(n)$
1	$0.a_1^1 a_2^1 a_3^1 a_4^1 a_5^1 \ldots$
2	$0.a_1^2 a_2^2 a_3^2 a_4^2 a_5^2 \ldots$
3	$0.a_1^3 a_2^3 a_3^3 a_4^3 a_5^3 \ldots$
4	$0.a_1^4 a_2^4 a_3^4 a_4^4 a_5^4 \ldots$
5	$0.a_1^5 a_2^5 a_3^5 a_4^5 a_5^5 \ldots$
$\vdots$	

[5] This is a bit of a pain to TEX, so you'll have to use your imagination. Sorry.
[6] At first.
[7] Exercise: Show that $|(0,1)| = |\mathbb{R}|$.
[8] This word is sweeping some rather massive insights under the rug. . . .
[9] Namely, $0.\overline{9} = 1$ and similar situations where the same real number can have two distinct decimal expansions.

Note that here, each $a_j^i \in \{0, 1, 2, \ldots, 0\}$ is simply a decimal, with two indices indicating where it sits in the array. Now, consider the decimal $x = 0.x_1x_2x_3x_4x_5\ldots$, where

$$x_i = \begin{cases} 0 & a_i^i \neq 0 \\ 1 & a_i^i = 0 \end{cases}.$$

Note that then $x \in (0, 1)$ has one of our acceptable decimal forms. I claim that x is nowhere on our list of real numbers that are hit by f. If not, then $x = f(n)$ for some n. But the nth decimal place of x is different from the nth decimal of $f(n)$ by construction, so in fact we have $x \neq f(n)$. In fact, x was constructed in such a way as to never appear on our list of real numbers, so we see that f cannot be surjective, as claimed. □

In light of this discovery that there are different sizes of infinity, the following definition is necessary.

Definition 9. A set A is *countable* (or *countably infinite*) if $|A| = |\mathbb{N}|$. If B is an infinite set such that $|B| \neq |\mathbb{N}|$, we say B is *uncountable.*

We'd like to have an analogue of natural numbers so we can write $|\mathbb{N}|$ = some symbol.

Notation 10. $|\mathbb{N}| = \aleph_0$.

$\aleph$ is the Hebrew letter "aleph," and the subscript 0 indicates that this is the *smallest infinite cardinal.* We've already seen that $|\mathbb{R}| > \aleph_0$, though whether the cardinality of $\mathbb{R}$ should be called $\aleph_1$ is formally undecidable. In other words, the question of whether there are infinite cardinals between α_0 and $|\mathbb{R}|$ is independent of the "standard" axioms of set theory, and so either choice may be selected as an axiom in its own right.