Discrete Mathematics lecture notes 5-1

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15. Relations

We begin with a fairly unmotivated definition:

Definition 1. If A and B are sets, a *relation from A to B* is a subset $R \subseteq A \times B$. If $(a, b) \in R$, we write $a \; R \; b$ to indicate this.

Example 2. Let $A = B = \mathbb{N}$.

- The successor relation $S \subseteq \mathbb{N} \times \mathbb{N}$ is defined by $a \; S \; b$ iff $b = a + 1$.
- The additive (total) order relation $T \subseteq \mathbb{N} \times \mathbb{N}$ is defined by $a \; T \; b$ iff $a \leq b$.
- The multiplicative (partial) order relation $P \subseteq \mathbb{N} \times \mathbb{N}$ is defined by $a \; P \; b$ iff $a|b$.

The last two examples show where the term "relation" comes from: We're weakening the notion of a partial order relation, which was in turn a weakening of the notion of total order relation. Of course, not all relations are orderings, partial or otherwise. There are certain properties[1] of relations that crop up often enough to warrant special terminology.

Definition 3. Let S be a set and R a relation on S.

- R is *reflexive* if $a \; R \; a$ for all $a \in S$.
- R is *irreflexive* if $a \not R \; a$ for all $a \in S$.
- R is *symmetric* if $a \; R \; b$ implies $b \; R \; a$ for all $a, b \in S$.
- R is *antisymmetric* if $a \; R \; b$ and $b \; R \; a$ imply $a = b$ for all $a, b \in S$.
- R is *transitive* if $a \; R \; b$ and $b \; R \; c$ implies $a \; R \; c$ for all $a, b, c \in S$.
- R is *total*[2] if $a \; R \; b$ or $b \; R \; a$[3] for all $a, b \in S$.

Exercise. Play around with relations to see what combinations of the above terms are possible. For instance, let S be the set of people and R some human relation ("is a child of," "shares a common cousin," "met once in high school," ...) and see what properties you can realize.

Exercise. Show that a partial ordering on S is precisely a relation that is reflexive, antisymmetric, and transitive, while a total ordering on S is precisely a partial ordering that is also total.

Small aside: There is actually an algebra of relations. The starting point for this realization is the following definition:

Definition 4. Let R be a relation from A to B, so $R \subseteq A \times B$. The *inverse of* R is the relation $R^{-1} \subseteq B \times A$ defined by $(b, a) \in R^{-1}$ iff $(a, b) \in R$.

Definition 5. If R is a relation from A to B and S is a relation from B to C, then $S \circ R$ is the *composite relation*, defined by $(a, c) \in S \circ R$ iff there is some $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

We'll come back to this notion later on.

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[1] This is as good place as any to introduce a recurring metaconcept: The relationship between a *property* and *structure*. Roughly speaking: A property is some condition imposed on an object of study, while structure is some additional level of data. For instance, the set $\mathbb{N}$ together with the *structure* of the partial ordering P is what makes $\mathbb{N}$ into a poset, while it is the *properties* of P (that is reflexive, antisymmetric, and transitive) that makes it into a partial ordering.

[2] Not standard terminology

[3] Or both: We always use the word "or" in the Boolean sense of $\vee$.

16. Functions

And now we come to the main idea. We want to relate various mathematical objects to one another, whether those objects are sets, or posets, or rings like $\mathbb{Z}$ or $\mathbb{Z}/n$, or .... The natural context for this study is the notion of function.

Definition 6. Let A and B be sets. A *function from A to B* is a relation $f \subseteq A \times B$ such that for all $a \in A$ there exists a unique $b \in B$ with $(a, b) \in f$. For this pair, we write as shorthand $f(a) = b$.

Example 7. The main object of study in calculus is functions from $\mathbb{R}$[4] to $\mathbb{R}$. If we think about the graphical representation of a function—the *graph* of the function—that is visibly given to us a a subset of $\mathbb{R} \times \mathbb{R}$. Checking that any random squiggle you could draw in the plane actually determines a function is equivalent to the famous *vertical line test*. Convince yourself of this.

Definition 8. If f is a function from A to B, we'll write $f : A \to B$. In this case, A is the *domain* or *source* of f, and B is the *range* or *target*.

Note that this definition of range only takes into account the set B, not the particular of the function $f : A \to B$. In particular, if $b_0 \in B$ is some chosen element, then we could define a *constant function* $c_{b_0} : A \to B : a \mapsto b_0$. Unless $B = \{b_0\}$, i.e., B is a singleton set, then there are elements of B that are never "hit" by f. Thus the notion of range that we're defining here may be slightly different from what you've encountered in earlier classes. To account for this discrepency:

Definition 9. Let $f : A \to B$ be a function. The *image of f* is $im(f) := \{b \in B | \exists\ a \in a \text{ such that } f(a) = b\}$.

Example 10. If $f : \mathbb{R} \to \mathbb{R}$ is the function $x \mapsto x^2$, then the domain of f is $\mathbb{R}$, the range of f is $\mathbb{R}$,[5] and the image of f is $\mathbb{R}_{\geq 0}$.

If $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is the function $x \mapsto x^2$, then even though $f(x) = g(x)$ for all $x \in dom(f) = dom(g)$, because f and g are defined as subsets of different sets, they are not equal.

One of the most important functions of functions is that they allow us to identify when sets are "the same,"[6] suitably understood. We'll deal with this a lot more later, but for now: If $f : A \to B$ is a function, then it is a *relation* (with the additional *property* that for all $a \in A$, there is a unique $b \in B$ such that $f(a) = b$), so we can look at the inverse relation f^{-1} defined before. $f^{-1} \subseteq B \times A$ may or may not be a function, depending on what exactly f is.

Definition 11. Let $f : A \to B$ be a function.

- f is *surjective*, or *onto*, if $im(f) = B$, i.e., for all $b \in B$, there is some[7] $a \in A$ such that $f(a) = b$.
- f is *injective*, or *into*, or *one-to-one*, if whenever we have $a, a' \in A$ such that $f(a) = f(a')$, it follows that $a = a'$.
- f is *bijective* if f is surjective and injective.

Theorem 12. *$f : A \to B$ a function. Then the relation f^{-1} from B to A is a function iff f is bijective.*

Proof. That f^{-1} is a function means for all $b \in B$ there is a unique $a \in A$ such that $(b, a) \in f^{-1}$. By definition of the inverse relation, this means for all $b \in B$ there is a unique $a \in A$ such that $f(a) = b$. Saying "for all $b \in B$ there is an $a \in A$ such that $f(a) = b$" is the definition of f's being surjective; saying that the element $a \in A$ such that $f(a) = b$ is unique is the definition of f's being injective. $\square$

[4]Or some subset of $\mathbb{R}$.
[5]Note: Not $\mathbb{R}_{\geq 0}$.
[6]Note: Not "equal."
[7]Not necessarily unique.