Problem 1 (10 pts.)

Consider a system with some Hamiltonian

H = H(p,x),

obeying Hamilton`s equations of motion (Sethna, Eq. 4.1). Is the distribution

$\rho(p,x)=A\exp(-\beta H(p,x))$

with A and β constants a valid equilibrium distribution? Explain your answer showing the details of the derivation.

Equilibrium distribution must be:

* stationary $\left(\frac{\partial \rho}{\partial t}=0\right) \Rightarrow$ TRUE
* satisfy Liouville $\Rightarrow \frac{d\rho}{dt}=0$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial \rho}{\partial x}\frac{\partial x}{\partial t}$$

$$= A\exp(-\beta H)\left(-\beta\frac{\partial H}{\partial p}\right)\frac{\partial p}{\partial t} + A\exp(-\beta H)\left(\frac{\partial H}{\partial x}\right)(-\beta)\frac{\partial x}{\partial t}$$

$$= -A\beta\exp(-\beta H)\left[\frac{\partial H}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial H}{\partial x}\frac{\partial x}{\partial t}\right]$$

using Sethna Eq. 4.1

$$= -A\beta\exp(-\beta H)\left[\frac{\partial x}{\partial t}\frac{\partial p}{\partial t} - \frac{\partial p}{\partial t}\frac{\partial x}{\partial t}\right] = 0$$ ✓

Problem 2 (10 pts.)

Sethna, problem 4.2.

for Hamiltonian dynamics:

$$\dot{p} = -\frac{\partial H}{\partial q} \qquad \dot{x} = \frac{\partial H}{\partial p}$$

$$\Rightarrow \frac{\partial \dot{p}}{\partial p} = -\frac{\partial H}{\partial p \partial x}$$

$$\frac{\partial \dot{x}}{\partial x} = \frac{\partial H}{\partial p \partial x}$$

$$\frac{\partial \dot{p}}{\partial p} = -\frac{\partial \dot{x}}{\partial x}$$

a.) $\frac{\partial \dot{p}}{\partial p} = -\gamma \qquad \frac{\partial \dot{x}}{\partial x} = 0$

~~they are not equal.~~

$\Rightarrow$ here this is not true $\Rightarrow$

(b) $$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \rho}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial \rho}{\partial t}$$

continuity eq'n.: $$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho \dot{x}) - \frac{\partial}{\partial p}(\rho \dot{p})$$

$$= -\frac{\partial \rho}{\partial x}\dot{x} - \rho\frac{\partial \dot{x}}{\partial x} - \frac{\partial \rho}{\partial p}\dot{p} - \rho\frac{\partial \dot{p}}{\partial p}$$

$$\frac{d\rho}{dt} = -\rho\frac{\partial \dot{x}}{\partial x} - \rho\frac{\partial \dot{p}}{\partial p} = +\rho\gamma$$

$$\Rightarrow \rho(t) = e^{\gamma t}\rho(0)$$

exponential increase since all pts. converse together

Problem 3(10 pts.)

Sethna, Problem 5.10.

$$S = -k_B \int dx\, \rho \ln \rho$$

$$\frac{dS}{dt} = -k_B \int dx \left[\dot\rho \ln \rho + \dot\rho\right]$$

$$= -k_B \int dx\, \dot\rho \left[\ln \rho + 1\right]$$

$$= -k_B \int dx\, D\rho'' \left[\ln \rho + 1\right]$$

$$= -Dk_B \int dx\, \rho'' \left[\ln \rho + 1\right]$$

$$= -Dk_B \left[\underbrace{\rho' \left[\ln \rho + 1\right]\Big|}_{0} - \int \frac{\rho'^2}{\rho} dx \right]$$

$$= Dk_B \int \frac{\rho'^2}{\rho} dx \geq 0$$

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can also use $\rho(x,t) = \exp\left(-\frac{x^2}{4Dt}\right) \frac{1}{\sqrt{4D\pi t}}$
