Problem 1 (Sethna 3.5):

(a) $\left[A-\pi(2r)^2(N-1)\right]^N \rightarrow$ configuration part

$\Rightarrow$ momentum part: $\mu = \dfrac{\pi^N R^{2N}}{N!}$ (eq'n. 3.13 for 2N)

$R = \sqrt{2mE}$

$$\mu = \frac{\pi^N (2mE)^N}{N!}$$

$\Rightarrow$ surface area of sphere: $\dfrac{d\mu}{dR} = 2N\pi^N (2mE)^{N-\frac{1}{2}} \xrightarrow[N\gg 1/2]{} 2N\pi^N (2mE)^N$

$$\Omega(E) = \frac{\left[A-\pi(2r)^2(N-1)\right]^N}{N!\, h^{2N}} \frac{2N\pi^N(2mE)^N}{N!}$$

$$S(E) = k\ln\Omega = [N\ln[A-N\pi(2r)^2] - N\ln N + N + \ln 2N + N\ln\pi + N\ln 2mE - 2N\ln h - N\ln N + N]k$$

$$\cancel{= N\ln\left[\frac{A}{2N} - \frac{\pi(2r)^2}{2}\right]}$$

$$= \left[N\ln\left[\frac{A}{N} - \pi(2r)^2\right] + 2N + N\ln\left(\frac{2mE\pi}{Nh^2}\right) + \cancel{\ln 2N}\right]k$$

$\ln 2N$ small as $N \to \infty$

$$S(E) = 2Nk + N\ln[a-b]\left[\frac{2m\pi E}{h^2 N}\right]$$

$a = \dfrac{A}{N}$, $b = \pi(2r)^2$

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configurational entropy: $k\ln\Omega(E) = Nk\ln\left[A-\pi(2r)^2(N-1)\right] - N(\ln N)k + Nk$

$$S_{conf} = Nk\ln[a-b] + Nk$$

$b = \pi r^2$, $a = A/N$

pressure: $TdS = dE + pdV - \mu dN$, $V \to A$

$$\frac{P}{T} = \left.\frac{\partial S}{\partial V}\right)_{E,N} \Rightarrow \frac{P}{T} = \frac{Nk}{(a-b)}\frac{1}{N} \Rightarrow P(a-b) = kT$$

$$P(A-Nb) = NkT$$

as $b \to 0 \Rightarrow PA = NkT$ (ideal gas law!)

Problem 2 (Sethna 3.8):

a.) Eq'n. 3.29 $\frac{\partial S}{\partial E} = \frac{1}{T}$

$$\frac{1}{k_B}\frac{\partial^2 S}{\partial E^2} = -\frac{1}{k_B T^2}\frac{\partial T}{\partial E} = -\frac{1}{k_B T}\frac{1}{N C_V T}$$

b.) $$\sigma_{E_1}^2 = -\frac{k_B}{\frac{\partial^2 S_1}{\partial E_1^2} + \frac{\partial^2 S_2}{\partial E_2^2}} = +\frac{k_B}{k_B\left(\frac{1}{NT^2}\right)\left(\frac{1}{C_V^{(1)}} + \frac{1}{C_V^{(1)}}\right)}$$

$$\rightarrow \quad = \frac{N k_B T^2}{\frac{1}{C_V^{(1)}} + \frac{1}{C_V^{(2)}}} \Rightarrow \frac{1}{C_V^{(1)}} + \frac{1}{C_V^{(2)}} = \frac{N k_B T^2}{\sigma_{E_1}^2}$$

c.) equipartition thm.: $\frac{K}{N} = \frac{3 k_B T}{2}$

$$C_V^{(1)} = \frac{\partial (K/N)}{\partial T} = \frac{3 k_B}{2}$$

$$\Downarrow$$

$$\frac{2}{3k_B} + \frac{1}{C_V^{(2)}} = \frac{N k_B T^2}{\sigma_K^2} \Rightarrow \frac{1}{C_V^{(2)}} = \frac{N k_B T^2}{\sigma_K^2} - \frac{2}{3k_B} = \frac{3 N k_B^2 T^2 - 2\sigma_K^2}{3 k_B \sigma_K^2}$$

$$C_V^{(2)} = \frac{3 k_B \sigma_K^2}{3N k_B^2 T^2 - 2\sigma_K^2}$$

total: $$C_V = C_V^{(1)} + C_V^{(2)} = \frac{3k_B}{2} + \frac{3 k_B \sigma_K^2}{3N k_B^2 T^2 - 2\sigma_K^2}$$

Problem 3:

Consider a system of L lattice sites. At each lattice site there is a spin which can be either up (spin s=½) or down (spin s=-½). Under an external magnetic field B, the energy of this system is given by

$$E = -\sum_{i=1} \mu B s$$

where μ is a constant.

1. Calculate the entropy, temperature, specific heat of this system in the microcanonical ensemble.

$$M \Rightarrow 1/2 \qquad L-M \to -1/2$$

$$E = -\sum_{i=1}^{L} \mu B s =$$

$$E(M) = -\mu B M + \mu B(L-M) = \mu B(L-2M)$$

$$\frac{E}{\mu B} = L - 2M \Rightarrow M = \frac{L}{2} - \frac{E}{2\mu B}$$

$$\Omega(E) = \frac{L!}{M!(L-M)!}$$

$$S(M) = k\left(L\ln L - M\ln M - (L-M)\ln(L-M)\right)$$

$$\boxed{S(E) = k\left[L\ln L - \left(\frac{L}{2} - \frac{E}{2\mu B}\right)\ln\left(\frac{L}{2} - \frac{E}{2\mu B}\right) - \left(\frac{L}{2} + \frac{E}{2\mu B}\right)\ln\left(\frac{L}{2} + \frac{E}{2\mu B}\right)\right]}$$

$$\frac{\partial S}{\partial E} = \frac{1}{T} \Rightarrow \frac{1}{kT} = \frac{1}{2\mu B}\ln\left(\frac{L}{2} - \frac{E}{2\mu B}\right) + \cancel{\frac{1}{2\mu B}} - \frac{1}{2\mu B}\ln\left(\frac{L}{2} + \frac{E}{2\mu B}\right) - \cancel{\frac{1}{2\mu B}}$$

$$e^{\frac{2\mu B}{kT}} = \frac{\frac{\mu B L}{2} - E}{\frac{\mu B L}{2} + E} \Rightarrow e^{2\mu B/kT}(\mu B L + E) = \mu B L - E$$

$$E\left(1 + e^{2\mu B/kT}\right) = \mu B L\left(1 - e^{2\mu B/kT}\right)$$

$$\boxed{T = \frac{2\mu B}{k\ln\left[\frac{\mu B L - E}{\mu B L + E}\right]}}$$

$$\boxed{E = \mu B L \tanh\left(\frac{\mu B}{kT}\right)}$$

$$\boxed{C = \frac{\partial E}{\partial T} = -\frac{(\mu B)^2 L}{kT^2}\left[1 - \tanh^2\left(\frac{\mu B}{kT}\right)\right]}$$