Hour Exam 2

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Problem 1:

Consider a random walk in which the probability to go right is given by p.

a. Calculate the average displacement after N steps.

b. Calculate the mean square displacement after N steps.

c. For the average displacement and the mean square displacement investigate the two limits

i. $p \to ½, N \to \infty$,

ii. $p \to 1, N \to \infty$.

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a.) $m_L \to$ moves to left

$m_R \to$ moves to right

$m \to$ displacement

$$m_L + m_R = N$$
$$m_R - m_L = m$$
$$m_R = \frac{N+m}{2}, \quad m_L = \frac{N-m}{2}$$

$$P(m) = p^{m_R} (1-p)^{m_L} \frac{N!}{m_L! \, m_R!} = p^{\frac{N+m}{2}} (1-p)^{\frac{N-m}{2}} \frac{N!}{\left(\frac{N+m}{2}\right)! \left(\frac{N-m}{2}\right)!}$$

$$\overline{m} = ? \to \sum_m P(m)\, m = \sum_{m=-N}^{N} p^{\frac{N+m}{2}} (1-p)^{\frac{N-m}{2}} \frac{N! \, m}{\left(\frac{N+m}{2}\right)! \left(\frac{N-m}{2}\right)!}$$

$m = -N, -N+2, \ldots, N-2, N$

calculate $\overline{\frac{N+m}{2}} = \frac{N}{2} + \frac{\overline{m}}{2}$

$$\left[\frac{N}{2} + \frac{\overline{m}}{2}\right] = \sum_{m=-N}^{N} p^{\frac{N+m}{2}} (1-p)^{\frac{N-m}{2}} \frac{N!}{\left(\frac{N+m}{2}\right)! \left(\frac{N-m}{2}\right)!} \left(\frac{N+m}{2}\right)$$

$*$ $m = -N$ term is zero

$$= \sum_{m=-N+2}^{N} p^{\frac{N+m}{2}} (1-p)^{\frac{N-m}{2}} \frac{N!}{\left(\frac{N+m}{2} - 1\right)! \left(\frac{N-m}{2}\right)!}$$

Worksheet: (con't)

* replace m with $m' = m - 1$

$$\sum_{m'=-N+1}^{N-1} p^{\frac{N+m'+1}{2}} (1-p)^{\frac{N-m'-1}{2}} \frac{N!}{\left(\frac{N+m'+1}{2}-1\right)! \left(\frac{N-m'-1}{2}\right)!}$$

$$= Np \Rightarrow \frac{N}{2} + \frac{\bar{m}}{2} = Np \Rightarrow \boxed{\bar{m} = N(2p-1)}$$

6.) calculate: $\overline{\left(\frac{N+m}{2}\right)\left(\frac{N-m}{2}\right)} = \frac{N^2 - \overline{m^2}}{4}$

$$= \sum_{m=-N}^{N} p^{\frac{N+m}{2}} (1-p)^{\frac{N-m}{2}} \frac{N!}{\left(\frac{N+m}{2}\right)! \left(\frac{N-m}{2}\right)!} \frac{(N+m)}{2}\frac{(N-m)}{2}$$

$$= \sum_{m=-N+2}^{N-2} p^{\frac{N+m}{2}} (1-p)^{\frac{N-m}{2}} \frac{N!}{\left(\frac{N+m}{2}-1\right)! \left(\frac{N-m}{2}+1\right)!}$$

$$\frac{N^2 - \overline{m^2}}{4} = N(N-1)p(1-p) = \cancel{N^2 p} \; N^2 p - N^2 p^2 - Np + Np^2$$

$$\Rightarrow \overline{m^2} - \bar{m}^2 = \cancel{N^2 p - N^2 p^2 - Np + Np^2} \; \cancel{N^2} - \cancel{4}N^2 p + \cancel{4N^2 p^2} + 4Np - 4Np^2 - \cancel{4N^2 p^2} + \cancel{4N^2 p} - \cancel{N^2}$$

$$\boxed{\overline{m^2} - \bar{m}^2 = \underline{4Np(1-p)}}$$

c.) limits: $p \to 1/2$ $\quad \bar{m} \to 0 \quad \cancel{4Np} \quad \overline{m^2} - \bar{m}^2 \to N$

$$\ln P(m) \to \cancel{\tfrac{1}{2}} - N\ln 2 + N\ln N - \left(\frac{N+m}{2}\right)\ln\left(\frac{N+m}{2}\right) - \left(\frac{N-m}{2}\right)\ln\left(\frac{N-m}{2}\right)$$

1st contributing term for small $m \Rightarrow$ ~~2nd~~ order

$$P(m) \sim \underline{\underline{\exp\left(-\frac{m^2}{N}\right)}} \quad \underline{\text{Gaussian}}$$

$p \to 1$ ~~$P(m)$~~ $m \to N \Rightarrow P(m) \sim p^{\frac{N+m}{2}} e^{-p\left(\frac{N-m}{2}\right)} \frac{1}{\left(\frac{N-m}{2}\right)!} N^{\left(\frac{N-m}{2}\right)}$

let $\frac{N-m}{2} = \ell \quad P(m) \sim \frac{e^{-\ell p}}{\ell!} N^{\ell} \Rightarrow \underline{\text{Poisson}}$

4 not 2 (typo)

Problem 2:

Consider a one-dimensional system undergoing diffusion according to the diffusion equation: $\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2}$. Given that at time τ the probability distribution is given by $\rho(x,\tau) = \frac{1}{\sqrt{2\pi D\tau}}\exp\left(-\frac{x^2}{4D\tau}\right)$, calculate the probability density at time $\tau+\sigma$. Show the calculation explicitly.

$$\rho(x,\tau+\sigma) = \int dy\, \rho(x-y;\sigma)\, \rho(y,\tau)$$

$$= \frac{1}{4\pi D\sqrt{\tau\sigma}} \int dy\, \exp\Big[\underbrace{-\frac{(x-y)^2}{4D\sigma} - \frac{y^2}{4D\tau}}\Big]$$

$$\frac{(x-y)^2}{4D\sigma} + \frac{y^2}{4D\tau} = \frac{\tau(x-y)^2+\sigma y^2}{4D\sigma\tau} = \frac{\tau x^2 - 2\tau xy + \tau y^2 + \sigma y^2}{4D\sigma\tau}$$

$$= \frac{x^2}{4D\sigma} + \frac{(\sigma+\tau)y^2 - 2y(x\tau)}{4D\sigma\tau}$$

$$= \frac{x^2}{4D\sigma} + \frac{(\sigma+\tau)\left[y^2 - \frac{2y(x\tau)}{(\sigma+\tau)} + \frac{x^2\tau^2}{(\sigma+\tau)^2} - \frac{x^2\tau^2}{(\sigma+\tau)^2}\right]}{4D\sigma\tau}$$

$$= \frac{x^2}{4D\sigma} - \frac{x^2\tau}{4D\sigma(\sigma+\tau)} + \frac{(\sigma+\tau)}{4D\sigma\tau}\left[y - \frac{x\tau}{\sigma+\tau}\right]^2$$

$$\underbrace{= \frac{x^2}{4D(\sigma+\tau)} + \frac{(\sigma+\tau)}{4D\sigma\tau}\left[y - \frac{x\tau}{\sigma+\tau}\right]^2}$$

$$\rho(x,\tau+\sigma) = \frac{1}{4\pi D\sqrt{\tau\sigma}} \exp\left[-\frac{x^2}{4D(\sigma+\tau)}\right] \underbrace{\int dy\, \exp\left[-\frac{(\sigma+\tau)}{4D\sigma\tau}\left(y-\frac{x\tau}{\sigma+\tau}\right)^2\right]}_{=\sqrt{\frac{4D\sigma\tau\pi}{\sigma+\tau}}}$$

$$\rho(x,\tau+\sigma) = \frac{1}{\sqrt{4\pi D(\sigma+\tau)}} \exp\left[-\frac{x^2}{4D(\sigma+\tau)}\right]$$

(can be done by Fourier transform also)

Problem 3 (Sethna 2.8):

The rate of energy flow in a material with thermal conductivity k_t and a temperature field $T(x, y, z, t) = T(\mathbf{r}, t)$ is $\mathbf{J} = -k_t \nabla T$. Energy is locally conserved, so the energy density E satisfies $\frac{\partial E}{\partial t} = -\nabla \cdot \mathbf{J}$.

1. If the material has constant specific heat c_p and density ρ, so $E = c_p \rho T$, show that the temperature T satisfie~~s~~d the diffusion equation.
2. By putting our material in a cavity with standing microwaves, we heat it with a periodic modulation $T = \sin(kx)$ at $t = 0$, at which time the microwaves are turned off. Show that the amplitude of the temperature modulation decays exponentially in time. How does the amplitude decay rate depend on wavelength $\lambda = 2\pi/k$?

1.) $$\frac{\partial E}{\partial t} = -\nabla \cdot \vec{J} \Rightarrow \frac{\partial E}{\partial t} = (\nabla^2 T)\, k_t \Rightarrow \boxed{\frac{\partial T}{\partial t} = \frac{k_t}{c_p \rho} \nabla^2 T}$$

2.) $$T(0) = \sin kx \Rightarrow T(t) = \underbrace{A(t)}_{\text{amplitude}} \sin kx$$

$$\frac{\partial T}{\partial t} = \frac{k_t}{c_p \rho} (-k^2 \sin kx)\, A(t) = \frac{\partial A(t)}{\partial t} \sin kx$$

$$\Rightarrow \underline{A(t) = \exp\left(-\frac{k_t}{c_p \rho} k^2 t\right) A(0)}$$

wavelength dependence:

$$\boxed{A(t) = \exp\left[-\frac{k_t}{c_p \rho} \frac{(2\pi)^2}{\lambda^2} t\right] A(0)}$$