Bose-Einstein Condensation (BEC)

occurs when $z = 1 \Rightarrow \mu = 0$

$$\underbrace{\langle n\rangle \lambda_T^3 = g_{3/2}(1)}\approx 2.612$$

determines phase diagram: * for a given temperature T density at which BEC occurs is $\langle n\rangle_c \lambda_T^3 = g_{3/2}(1)$

* for a given density $\langle n\rangle$, temperature at which BEC occurs is $\langle n\rangle \lambda_{T_c}^3 = g_{3/2}(1)$

$$\Downarrow$$

$$\langle n\rangle \left(\frac{2\pi\hbar^2}{m k_B T_c}\right)^{3/2} = g_{3/2}(1)$$

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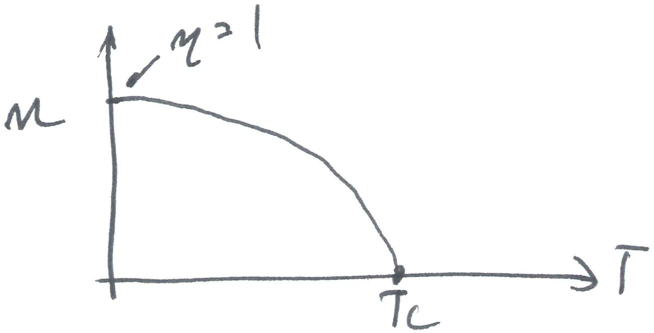
below T_c : $\langle n\rangle = n_0 + \frac{g_{3/2}(1)}{\lambda_T^3}$

$$\langle n\rangle = n_0 + \langle n\rangle \left(\frac{T}{T_c}\right)^{3/2}$$

$$\boxed{\eta = \frac{n_0}{\langle n\rangle} = 1 - \left(\frac{T}{T_c}\right)^{3/2}}$$

$\eta \rightarrow$ BEC fraction $\rightarrow$ order parameter



at $T=0 \Rightarrow \langle n\rangle = n_0$ whole system is BEC determine P-V coexistence curve

eliminate T:

$$P = \frac{k_B T}{\lambda_T^3} g_{5/2}(1) = \frac{k_B (m k_B)^{3/2}}{(2\pi\hbar^2)^{3/2}} T^{5/2} g_{5/2}(1)$$

$$\langle n \rangle = \frac{(m k_B)^{3/2} T^{3/2}}{(2\pi\hbar)^{3/2}} g_{3/2}(1)$$

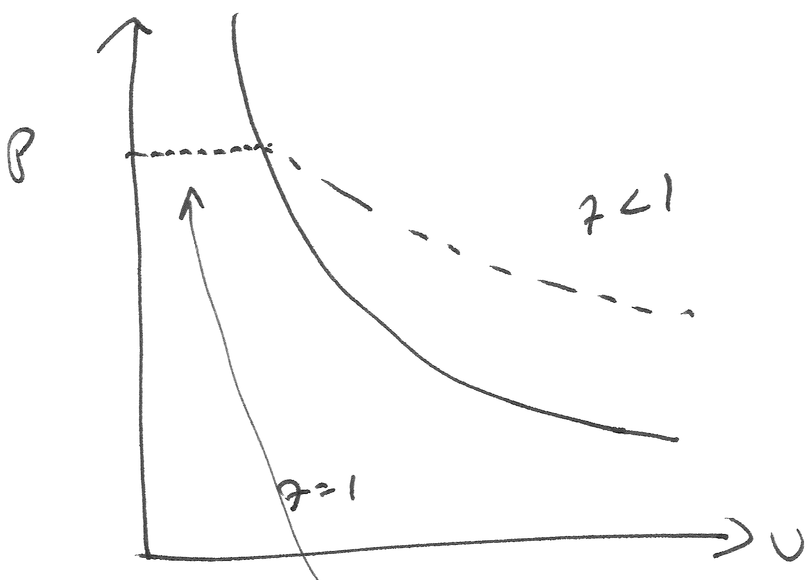
$$T^{3/2} = \frac{(2\pi\hbar^2)^{3/2}}{(m k_B)^{3/2}} \frac{1}{g_{3/2}(1)} \langle n \rangle$$

$$T = \frac{(2\pi\hbar^2)}{(m k_B)} \frac{\langle n \rangle^{2/3}}{[g_{3/2}(1)]^{2/3}}$$

$$P = \frac{k_B (m k_B)^{3/2}}{(2\pi\hbar^2)^{3/2}} \frac{(2\pi\hbar^2)^{5/2}}{(m k_B)^{5/2}} \frac{\langle n \rangle^{5/3}}{[g_{3/2}(1)]^{5/3}} g_{5/2}(1)$$

$\Downarrow$

$$P_c \sim \frac{1}{V_c^{5/3}}$$



for isotherms

$\left.\frac{\partial P}{\partial V}\right)_T \leq 0 \Rightarrow$ stability

in the BEC region

$(z=1)$

$$P = \frac{k_B T}{\lambda_T^3} g_{5/2}(1)$$

$\Downarrow$

P does not depend on V $\Rightarrow$ constant calculate heat capacity / entropy

$$d\Omega = -S dT - P dV$$

$$\Rightarrow s = \frac{\partial S}{\partial V}\Big)_{T,\mu} = \frac{\partial P}{\partial T}\Big)_{V,\mu}$$

$$s = \frac{5}{2}\frac{k_B}{\lambda_T^{3/2}} g_{5/2}(z) + \frac{k_B T}{\lambda_T^3} g'_{5/2}(z)\frac{\partial z}{\partial T} \qquad \boxed{z < 1}$$

$$\frac{\partial z}{\partial T} = \frac{\partial e^{\beta\mu}}{\partial T} = e^{\beta\mu}\left(-\frac{1}{k_B T^2}\right) = -\frac{z\beta\mu}{T}$$

$$s = \frac{5}{2}\frac{k_B}{\lambda_T^{3/2}} g_{5/2}(z) - \frac{k_B}{\lambda_T^3} g'_{5/2}(z)\, z\beta\mu$$

$$s = \frac{5}{2}\frac{k_B}{\lambda_T^{3/2}} g_{5/2}(z) - \frac{k_B}{\lambda_T^3} g_{3/2}(z)\ln z$$

$$s = \begin{cases} \frac{5}{2}\frac{k_B}{\lambda_T^{3/2}} g_{5/2}(z) - k_B \langle n\rangle \ln z & z < 1 \\ \frac{5}{2}\frac{k_B}{\lambda_T^{3/2}} g_{5/2}(1) & z = 1 \end{cases}$$

$T\frac{\partial s}{\partial T}\Big)_n \Rightarrow$ need derivative under condition n - fixed

derivation will need $\frac{\partial z}{\partial T}$ with $\langle n\rangle$ fixed

$$\langle n\rangle = \frac{1}{\lambda_T^3} g_{3/2}(z)$$

$$0 = +\frac{3}{2}\frac{1}{\cancel{\lambda_T^3} T} g_{3/2}(z) + \frac{1}{\cancel{\lambda_T^3}} z g'_{3/2}(z)\frac{1}{z}\frac{\partial z}{\partial T}$$

$$\frac{3}{2T} g_{3/2}(z) + g_{1/2}(z)\frac{\partial}{\partial T}\ln z \Rightarrow \frac{\partial \ln z}{\partial T} = -\frac{3 g_{3/2}(z)}{2T g_{1/2}(z)}$$

$\underline{z<1}$ 12

$$\left.\frac{\partial s}{\partial T}\right)_n = k_B \frac{15}{4} \frac{1}{\lambda_T^3 T} g_{5/2}(z) + k_B \frac{5}{2} \frac{1}{\lambda_T^3} g'_{5/2}(z) \frac{\partial z}{\partial T}$$

$$- k_B \langle n\rangle \frac{\partial \ln z}{\partial T}$$

$$= k_B \frac{15}{4} \frac{1}{\lambda_T^3 T} g_{5/2}(z) + k_B \frac{5}{2} \frac{1}{\lambda_T^3} g_{3/2}(z) \frac{\partial \ln z}{\partial T}$$

$$- k_B \langle n\rangle \frac{\partial \ln z}{\partial T}$$

$$= k_B \left(\frac{15}{4}\right) \frac{1}{\lambda_T^3} \frac{1}{T} g_{5/2}(z) + k_B \frac{3}{2} \langle n\rangle \frac{\partial \ln z}{\partial T}$$

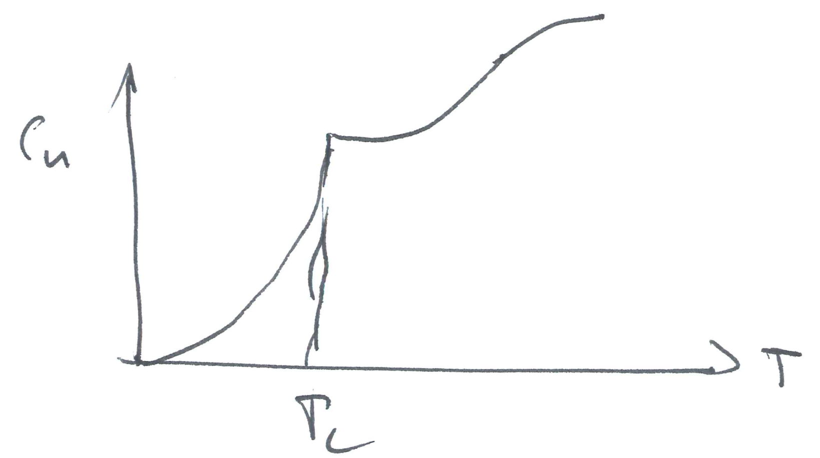
$$= k_B \left(\frac{15}{4}\right) \frac{1}{\lambda_T^3} \frac{1}{T} g_{5/2}(z) - k_B \frac{9}{4T} \langle n\rangle \frac{g_{3/2}(z)}{g_{1/2}(z)}$$

$$C_n = T \left.\frac{\partial s}{\partial T}\right)_n = \begin{cases} k_B \left(\frac{15}{4}\right) \frac{1}{\lambda_T^3} g_{5/2}(z) - k_B \frac{9}{4} \langle n\rangle \frac{g_{3/2}(z)}{g_{1/2}(z)} & z<1 \\ k_B \left(\frac{15}{4}\right) \frac{1}{\lambda_T^3} g_{5/2}(1) & z=1 \end{cases}$$

$s \to 0$ as $T \to 0$ $\Rightarrow$ obeys third law of thermodynamics

C_n

$\Rightarrow$



Ideal Fermi-Dirac Gas

- spin $s = 1/2$, mass m, box L

$$Z_{FD}(T,V,\mu) = \prod_{\vec{\ell}} \left(\sum_{n_{\vec{\ell}\uparrow}=0}^{1} e^{-\beta(\epsilon_{\vec{\ell}}-\mu)n_{\vec{\ell}\uparrow}} \right) \left(\sum_{n_{\vec{\ell}\downarrow}=0}^{1} e^{-\beta(\epsilon_{\vec{\ell}}-\mu)n_{\vec{\ell}\downarrow}} \right)$$

$$= \prod_{\vec{\ell}} \left(1 + e^{-\beta(\epsilon_{\vec{\ell}}-\mu)}\right)^2$$

- if spin is $3/2$ $\Rightarrow$ 4 states

$$Z_{FD}(T,V,\mu) = \prod_{\vec{\ell}} \left(1 + e^{-\beta(\epsilon_{\vec{\ell}}-\mu)}\right)^g$$

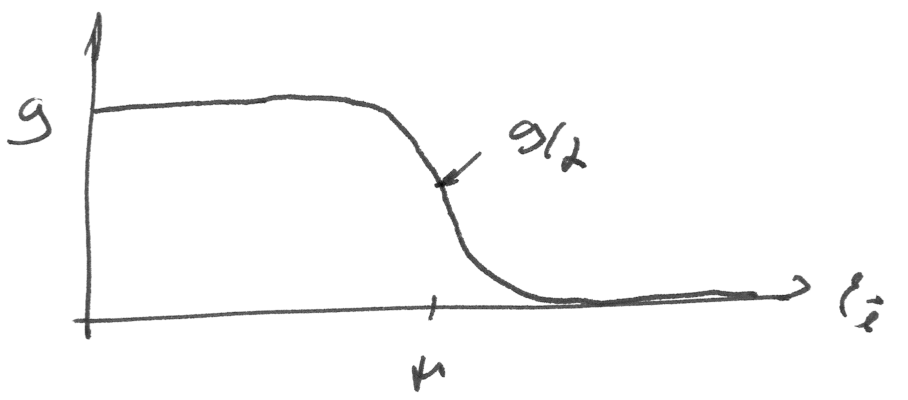
$$\Omega_{FD}(T,V,\mu) = -k_B T \ln Z_{FD}(T,V,\mu) = -k_B T g \sum_{\vec{\ell}} \ln\left(1 + e^{-\beta(\epsilon_{\vec{\ell}}-\mu)}\right)$$

$$\langle N \rangle = \frac{-\partial \Omega_{FD}(T,V,\mu)}{\partial \mu} = -k_B T g \sum_{\vec{\ell}} \frac{e^{-\beta(\epsilon_{\vec{\ell}}-\mu)}}{1 + e^{-\beta(\epsilon_{\vec{\ell}}-\mu)}} (-\beta)$$

$$= g \sum_{\vec{\ell}} \frac{z}{e^{\beta\epsilon_{\vec{\ell}}} + z}$$

$$\langle n_{\vec{\ell}} \rangle = \frac{gz}{e^{\beta\epsilon_{\vec{\ell}}} + z}$$

$\Rightarrow$ distribution:



change sum to integration

$$\Omega_{FD}(T,V,\mu) = -PV = -\frac{4\pi g k_B T V}{(2\pi\hbar)^3}\int_0^\infty p^2 dp \ln[1+z e^{-\beta p^2/2m}]$$

$$\langle N\rangle = \frac{4\pi g V}{(2\pi\hbar)^3}\int_0^\infty p^2 dp \left(\frac{z}{e^{\beta p^2/2m}+z}\right)$$

$$\Downarrow$$

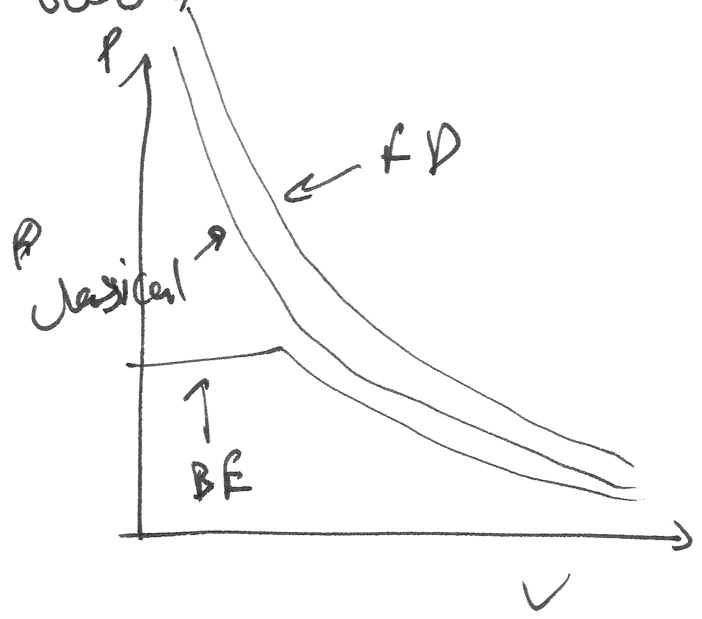
$$P = -\frac{\Omega_{FD}}{V} = \frac{g k_B T}{\lambda_T^3} f_{5/2}(z)$$

$$\langle n\rangle = \frac{\langle N\rangle}{V} = \frac{g}{\lambda_T^3} f_{3/2}(z)$$

$$f_{5/2}(z) = \frac{4}{\sqrt{\pi}}\int_0^\infty x^2 dx \ln[1+z e^{-x^2}] = \sum_{\alpha=1}^{\infty}(-1)^\alpha \frac{z^\alpha}{\alpha^{5/2}}$$

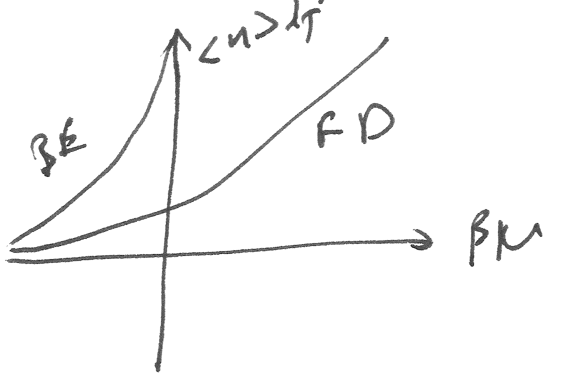
$$f_{3/2}(z) = z\frac{d}{dz} f_{5/2}(z) = \frac{4}{\sqrt{\pi}}\int_0^\infty x^2 dx \left(\frac{z}{e^{x^2}+z}\right) = \sum_{\alpha=1}^{\infty}(-1)^\alpha \frac{z^\alpha}{\alpha^{3/2}}$$

pressure vs. volume for ideal gases: classical, Boson, fermion



BE ⇒ condensate does not contribute to pressure

$\langle n\rangle \lambda_T^3$ vs. μ



BE $\mu \to 0$ $(T \to 0)$

$\Rightarrow \langle n\rangle \lambda_T^3 \to \infty$

FD $\mu \to$ finite

as $T \to 0$

invert series $\quad \dfrac{\langle n\rangle \lambda_T^3}{g} = f_{3/2}(z)$

$$f_{3/2}(z) = \sum_{\alpha=1}^{\infty} (-1)^{\alpha+1} \frac{z^\alpha}{\alpha^{3/2}}$$

$$z = \frac{\langle n\rangle \lambda_T^3}{g} + \frac{1}{2^{3/2}} \left(\frac{\langle n\rangle \lambda_T^3}{g} \right)^2 + \left(\frac{1}{2^2} - \frac{1}{3^{3/2}} \right) \left(\frac{\langle n\rangle \lambda_T^3}{g} \right)^3 + \ldots$$

as $T \to \infty$ $\quad z \to 0$ $\quad \beta\mu \Rightarrow$ large and negative

as $\beta \to 0$ $\quad \mu \to -\infty$

$\cancel{z \to \infty}$ low-T $\quad \Rightarrow z \to \infty$ $\quad \beta\mu \to \infty$ $\quad \beta \to \infty$

$\mu \to$ can remain finite

$$f_{3/2}(z) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx \frac{x^2}{(e^{x^2-\nu}+1)}$$

$y = x^2$

$x = \sqrt{y}$

$dx = \dfrac{dy}{2\sqrt{y}}$

$$= \frac{\cancel{4}^{\,2}}{\sqrt{\pi}} \int_0^\infty \frac{dy}{\cancel{2}\sqrt{y}} \frac{\sqrt{y}}{(e^{y-\nu}+1)}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{dy \sqrt{y}}{(e^{y-\nu}+1)} = \frac{4}{3\sqrt{\pi}} \int_0^\infty dy\, y^{3/2} \frac{e^{y-\nu}}{[e^{y-\nu}+1]^2}$$

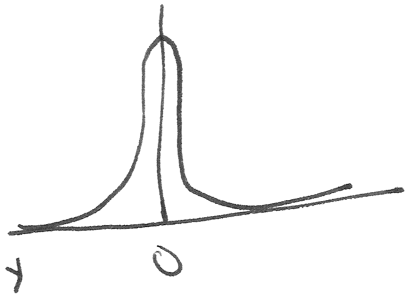
(integration by parts)

$t = \gamma - \nu$

$$\frac{4}{3\sqrt{\pi}} \int_{-\nu}^{\infty} dt\, t^{3/2} \frac{e^t}{(e^t+1)^2}$$

$$t^{3/2} = \nu^{3/2} + \frac{3}{2}\nu^{1/2} t + \frac{3}{8\nu^{1/2}} t^2 + \dots$$

$$\simeq \frac{4}{3\sqrt{\pi}} \int_{-\nu}^{\infty} dt\, \underbrace{\frac{e^t}{(e^t+1)^2}} \left[\nu^{3/2} + \frac{3}{2}\nu^{1/2} t + \frac{3}{8\nu^{1/2}} t^2 + \dots\right]$$



$\nu = \beta\mu$

$$\frac{e^t}{(e^t+1)^2} \to e^{-\beta\mu}$$

$\Downarrow$

can extend integral to $-\infty$

$$= \frac{4}{3\sqrt{\pi}} \int_{-\infty}^{\infty} dt\, \frac{e^t}{(e^t+1)^2} \left[\nu^{3/2} + \frac{3}{2}\nu^{1/2} t + \frac{3}{8\nu^{1/2}} t^2 + \dots\right]$$

$$I_n = \int_{-\infty}^{\infty} dt\, \frac{e^t}{(e^t+1)^2} t^n$$

$I_n = 0$ for n odd

$$I_n = \frac{(n-1)!}{(2\pi)(1-2^{1-n})} \zeta(n) \quad n \text{ even}$$

Riemann zeta function

$$\langle n\rangle \frac{\lambda_T^3}{g} = \frac{4}{3\sqrt{\pi}}\left[(\beta\mu)^{3/2} + \frac{\pi^2}{8}(\beta\mu)^{-1/2} + \ldots\right]$$ (17)

$$T \to 0 \Rightarrow \mu(T=0) = \frac{\hbar^2}{2m}\left(\frac{6\pi^2\langle n\rangle}{g}\right)^{2/3}$$

$\Downarrow$

Fermi energy

at $T = 0K$ Fermi energy is the maximum energy a particle can have

at low-T particles will be excited to on the order of $k_B T$

$\Rightarrow$ physics takes place near Fermi surface

$$\mu = \varepsilon_F\left(1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \ldots\right)$$

can also calculate internal energy

$$U = \frac{3}{5}\langle N\rangle \varepsilon_F\left(1 + \frac{5\pi^2}{12}\left(\frac{k_B T}{\varepsilon_F}\right)^2 + \ldots\right)$$

$$C_V = \frac{\langle N\rangle \pi^2}{2} \frac{k_B T}{\varepsilon_F}$$

specific heat is linear at low-T

$\Rightarrow$ usually in metals electronic degrees of freedom contribute linearly