renormalization group: general theory

(1)

- naive perturbation theory is divergent for $D \geq 4$
- better alternative: renormalization group theory

  - renormalization group: general theory (general framework) ⇒ has to be adapted to the problem at hand
    → involves guesswork, intuitive reasoning, mathematical and possibly numerical skills
- first give an intuitive picture
- a priori length scales ⇒ microscopic length scale a (lattice spacing) ⇒ corresponds to interaction length scale, if interactions are not long range
  ⇒ correlation length ξ

  → measure of distance over which spins are correlated

ξ - does not have a unique definition
some possible definition

$$\xi = \frac{\sum_j \langle |r_i - r_j| \, S_i S_j \rangle}{\sum_j \langle S_i S_j \rangle}$$

$$\xi \rightarrow \quad G(r) = \frac{\exp(-r/\xi)}{r^{D-2+\eta}}$$

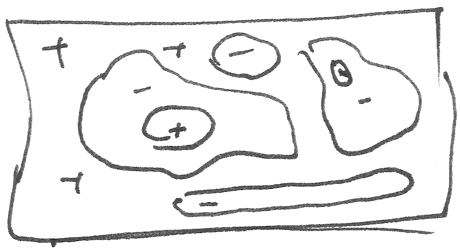
can integrate $G(r) \Rightarrow \int r^{D-1} dr \frac{e^{-r/\xi}}{r^{D-2+\eta}}$

we write ⇒ $\frac{\xi}{a} = \xi_{\pm} |t|^{-\nu}$ $t = \frac{T - T_c}{T_c}$

$\xi_+ \to T \to T_c+$ (from above)
$\xi_- \to T \to T_c-$ (from below)

at T_c spins display a fractal structure → in oceans of up spins there are islands of down spins with lakes of up spins

⇓



ξ → controls cluster size
near T_c ⇒ largest clusters are of extent ξ
at T_c $\xi \to \infty$ ⇒ all cluster sizes present ⇒ scale invariance

microscope analogy for renormalization group
consider two microscopes : A. has resolution a
B. has lower resolution b: $a \ll b \ll \xi$
look at system with microscope A near T_c
1.) if we look at system with microscope A near T_c we pick up all fluctuations
2.) if we look at system with microscope B near T_c some of the fluctuations are blurred ⇒ we would still observe clusters ξ/b ⇒ so the overall picture would be "similar" to that of microscope A
3.) if we reduce magnification of microscope B by b ⇒ we would see a system with correlation length ξ/b

- if resolution $b \sim \xi$ (bad resolution) picture would be of a system which is not close to T_c

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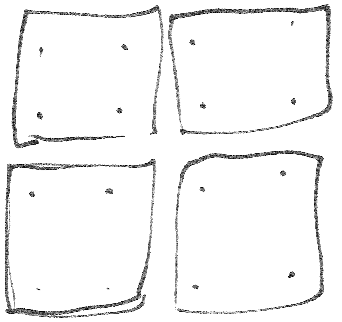
implement this picture mathematically: three steps

1.) <u>define spin blocks</u>



⇒ edge of block: <u>b</u>

$b \to$ scaling factor

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average: $S_I(b) = \frac{1}{b^D} \sum_{i \in I} S_i$

note: a.) if we keep doing blocking transformations ⇒ $S_I(b)$ becomes a continuous variable between $[+1, -1]$

b.) two blocking transformations can be written as one:

$$b'' = b b'$$

2.) <u>rescale unit length</u>: $L \to L/b$ ⇒ lattice is the same as before → can compare two systems

3.) <u>rescale spin blocks</u>: ~~to~~ if we compare two different renormalizations $S_I(b_1)$, $S_J(b_2)$ ⇒ probability distribution will be different in the two cases

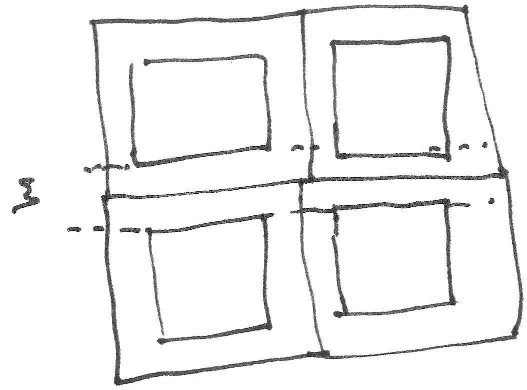
→ rescale blocks so that variations are similar ⇒ $\boxed{S_I(b) \to \varphi_I(b) = Z_1(b) S_I(b)}$

- consider case of high temperature:

* as renormalization transformations are applied $\Rightarrow$ $\xi \to \frac{\xi}{b} \to \frac{\xi}{b^2} \to \frac{\xi}{b^3} \cdots$

  eventually ξ for the renormalized system will tend to zero $\Rightarrow$ blocks will be nearly uncorrelated

* if correlation length is ξ $\Rightarrow$



spins participating in interaction between blocks

$$\Downarrow$$

$$\boxed{b^{D-1}\xi}$$

fraction of spins participating in interaction within a block $\Rightarrow$ $\frac{b^{D-1}\xi}{b^D} = \frac{\xi}{b} \xrightarrow[b\to\infty]{} 0$

* joint probability distribution of the blocks becomes the product of independent single-block distributions
  - spread of distribution controlled by susceptibility χ

$$\to \langle S_I^2(b)\rangle_c = \frac{\sum_{i\in I}\sum_{j\in I}\langle S_i S_j\rangle}{b^{2D}} = \frac{b^D\chi}{b^{2D}} = \underline{b^{-D}\chi}$$

5

$$P[S_I(b)] = \left(\frac{b^D}{2\pi\chi}\right)^{1/2} \exp\left(-\frac{b^D S_I^2(b)}{2\chi}\right)$$

choose rescaling as: $S_I(b) \to \bar{\varphi}_I(b) = b^{D/2} S_I(b)$
$= \bar{Z}_1(b) S_I(b)$

$$\bar{Z}_1(b) = b^{D/2}$$

$$\Rightarrow P_+[\bar{\varphi}_I(b)] = \frac{1}{\sqrt{2\pi\chi}} \exp\left[-\frac{\bar{\varphi}_I(b)}{2\chi}\right]$$

$\rightarrow$ fixed point of renormalization group ($b \to \infty$, high-temperature fixed point)

$T < T_c \rightarrow$ impose up-spins at the boundary

$\Downarrow$

in this case magnetization is finite

$$M = \langle S_i \rangle$$

$$\langle S_I^2(b) \rangle = \sum_{i\in I} \sum_{j \in I} \frac{\langle S_i S_j \rangle}{b^D}$$

$$= \sum_{i\in I} \sum_{j\in I} \frac{\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle + \langle S_i \rangle \langle S_j \rangle}{b^D}$$

$$= M^2 + \sum_{i\in I} \sum_{j\in I} \frac{\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle}{b^D}$$

$$\boxed{\langle S_I^2(b) \rangle = M^2 + b^{-D}\chi}$$

choose trivial rescaling $S_I(b) \to \bar{\varphi}_I(b) = b^0 S_I(b)$
$= \bar{Z}_I(b) S_I(b)$

$$\Rightarrow P_-[\bar{\varphi}_I(b)] = \frac{1}{\sqrt{2\pi\chi b^{-D}}} \exp\left[-\frac{(\bar{\varphi}_I(b) - M)^2}{2\chi b^{-D}}\right]$$

as $b \to \infty$ $\quad P_-[\bar{\varphi}(b)] \to \underline{\delta(\bar{\varphi}_I(b) - \bar{M})}$

low temperature fixed point

~~f~~

block scaling factors: $\boxed{\begin{array}{l} T > T_c: \ \bar{Z}_I(b) = b^{D/2} \\ T < T_c: \ \bar{Z}_I(b) = b^0 \end{array}}$

what about the critical temperature? ($T = T_c$

expect that $\Rightarrow$ b^{ω_1} scaling factor $\boxed{0 < \omega_1 < D/2}$

$$\boxed{\varphi_I(b) = Z_1(b)\, S_I(b) = b^{\omega_1} S_I(b)}$$

Critical exponents and scaling transformations

probability distributions derived above depend on temperature t and magnetization M

- can eliminate temperature dependence by considering two renormalization transformations $\Rightarrow b_1, b_2$ ~~such~~ chosen such that $\boxed{\frac{\xi_1}{b_1} = \frac{\xi_2}{b_2}}$

* let us do it for the high temperature case

$$\frac{\xi_1}{b_1} = \frac{\xi_2}{b_2} = c' \ll \text{lattice constant}$$

susceptibility : $\chi \sim \xi^{2-\eta}$

$$\Rightarrow \chi = c'' \xi^{2-\eta}$$

- define new scaling factor by (7)

$$Z_1(b_i) = \frac{b_i^{D/2}}{\sqrt{\chi}} = \frac{b_i^{D/2}}{c'' \xi_i^{\frac{2-\eta}{2}}} = \frac{b_i^{D/2}}{\sqrt{c''}\, c'^{\frac{2-\eta}{2}}\, b_i^{\frac{2-\eta}{2}}}$$

$$Z_1(b_i) = \frac{b_i^{\frac{D-2+\eta}{2}}}{\sqrt{c'' c'^{2-\eta}}} = c \; b_i^{\frac{D-2+\eta}{2}}$$

$\Downarrow$ $\Downarrow$ $\Downarrow$ $\Downarrow$

$$\boxed{P_+^*[\varphi(b_i)] = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\varphi^2(b_i)}{2}\right]}$$

→ in what follows we assume that this property is general: ⇒ if we start from two probability distributions at different temperatures, with different correlation lengths, and renormalize such that $\frac{\xi_1}{b_1} = \frac{\xi_2}{b_2}$ ~~then~~ then probability distribution of block spins will be identical, if the proper rescaling factor is used

$$\Rightarrow \quad P[S(b)] = Z_1(b)\, P_\pm^*\left[\frac{\xi}{b}, Z_1(b) S(b)\right]$$

↗ parameter of the probability distribution

→ this assumption has been verified via numerical simulations on Ising model

$\Downarrow$ assumption holds within universality class

- scaling factor in front of P → assures correct normalisation (8)

$$\int p(s)\,ds = \int P_\pm^*\left(\frac{\xi}{a}, Z_1 S\right) Z\,ds$$

$$p(s) = Z_1 P_\pm\left(\frac{\xi}{a}, Z_1 S\right)$$

- rewrite in a slightly more convenient form

$$\frac{\xi}{ba} = \xi_\pm |t|^{-\nu} b^{-1} = \left(\xi_\pm^{-1/\nu} |t| b^{1/\nu}\right)^{-\nu}$$

define $Z_2(b) = \xi_\pm^{-1/\nu} b^{1/\nu} = z_2^\pm b^{1/\nu}$

$$P[S(b)] = Z_1(b)\, P^*[Z_2(b)t;\ Z_1(b) S(b)]$$

$$= z_1^\pm b^{\frac{1}{2}(D-2+\eta)} P^*[z_2^\pm b^{1/\nu} t;\ z_1^\pm b^{\frac{1}{2}(D-2+\eta)} S(b)]$$

below T_c: $M = \langle S(b)\rangle = Z_1(b) \int dS(b)\, S(b)\, P[z_2^- b^{1/\nu} t, \varphi(b)]$

$$= Z_1(b) \int dS(b)\, S(b)\, P[z_2^- b^{1/\nu} t;\ z_1^- b^{\omega_1} S(b)]$$

$$= Z_1^{-1}(b) \underbrace{\int d\varphi(b)\, \varphi(b)\, P[z_2^- b^{1/\nu} t;\ \varphi(b)]}_{\text{depends on } z_2^- \underbrace{b^{1/\nu} t}\ \Rightarrow g(b^{1/\nu} t)}$$

$$M = Z_1^{-1}(b)\, g(b^{1/\nu} t)$$

since $M \sim t^\beta$

$$\Rightarrow M \simeq Z_1^{-1}(b)\, b^{\beta/\nu} t^\beta$$

$$\Rightarrow Z_1(b) \sim b^{+\beta/\nu} \Rightarrow Z_1(b) = z_1^- b^{\beta/\nu}$$

$$\Rightarrow \boxed{\begin{array}{ll} Z_1(b) = z_1^- b^{\omega_1} & \omega_1 = \beta/\nu \\ Z_2(b) = z_2^- b^{\omega_2} & \omega_2 = 1/\nu \end{array}}$$

are these results consistent? Check...

$$\langle S^2(b)\rangle = M^2 + b^{-D}\chi = M_0^2\, t^{2\beta} + \chi_0 b^{-D} t^{2-\alpha}$$
$$= \bar{M}_0^2\, b^{2\beta/\nu} + \bar{\chi}_0 b^{-D+2-\alpha}$$

for a temperature independent limit distribution the two contributions must have the same b-dependence

$$\Rightarrow \quad \frac{2\beta}{\nu} = -D+2-\alpha$$

$$\omega_1 = \frac{D-2+\alpha}{2}$$

$\Rightarrow$ results: $\Rightarrow$ ω_1 is the same for high-T and low-T limit distributions

$\Rightarrow$ critical exponents are not independent

Summary:

* within a universality class

$$P[S(b),t] = z_1 b^{\omega_1} P^*[z_2^{\pm} b^{\omega_2} t;\; z_1^{\pm} b^{\omega_1} S(b)]$$

- b must be large enough to average out details
- short range properties are included in $z_1^{\pm}, z_2^{\pm}$

* at T_c $\Rightarrow$ $P[S(b),0] = z_1 b^{\omega_1} P^*[0;\; z_1^{\pm} b^{\omega_1} S(b)]$

$$\Downarrow$$

$$P[S(b_1),0] = z_1 b_1^{\omega_1} P^*[0;\; z_1^{\pm} b_1^{\omega_1} S(b_1)]$$
$$P[S(b_2),0] = z_1 b_2^{\omega_1} P^*[0;\; z_1^{\pm} b_2^{\omega_1} S(b_2)]$$

$\rightarrow$ range of variation of $S(b_1)$ is rescaled compared to $S(b_2)$ by a factor $\left(\frac{b_2}{b_1}\right)^{\omega_1}$

* if $b_1^{\omega_1} t_1 = b_2^{\omega_2} t_2 \Rightarrow$ probability distributions (10)

at t_1, t_2 are identical provided

$$\left(\frac{b_1}{b_2}\right)^{\omega_1} = \left(\frac{t_2}{t_1}\right)^{\omega_2/\omega_1} = \left(\frac{t_2}{t_1}\right)^{\omega_1 \nu}$$

* joint probability distributions are also universal

$$P_2[S_I(b), S_J(b); t] =$$
$$= z_1^2\, b^{2\omega_1} P_2^*(z_2^t b^{\omega_2} t;\ z_1 b^{\omega_1} S_I(b);$$
$$z_1 b^{\omega_1} S_J(b);\ \bar{r}_I - \bar{r}_J]$$