Renormalization group theory (basic) 1

- rigorous foundation of Kadanoff scaling due to Wilson
- start with partition function of the system:

$$Z(K_1, K_2, \dots; N) = \sum_{S_1} \dots \sum_{S_N} e^{-\mathcal{H}(K_1, K_2 \dots ; S_1 \dots S_N)}$$

(note that β the inverse temperature has been absorbed in the variables $K_1, K_2 \dots K$)

- the Hamiltonian here is assumed to be more general than the usual Ising Hamiltonian

$$\mathcal{H}[K_1 \dots ; S_1 \dots S_N] = K_0 + K_1 \sum_i S_i + K_2 \sum_{\langle i,j \rangle} S_i S_j$$
$$+ K_2' \sum_{\langle\langle i,j \rangle\rangle} S_i S_j + \dots$$
$$+ K_3 \sum_{\langle ijk \rangle} S_i S_j S_k + K_3' \sum_{\langle ijk \rangle} S_i S_j S_k$$
$$+ \dots$$

- can include nearest neighbor pairs, next nearest neighbor pairs, etc. but also triplets, quadruplets, etc.

- introduce blocks: $S_i \to \tilde{S}_I$ (size b^D) ②

- blocking will give rise to a new Hamiltonian

$$H'[K'_1, \ldots; \tilde{S}_I \ldots]$$

- fix parameters of the block Hamiltonian by requiring that the <u>partition functions</u> of the blocked and unblocked Hamiltonians (and the relevant Boltzmann factors) are equal

$$\Rightarrow \sum_{\tilde{S}_I} \exp[-H'[K'_1 \ldots; \tilde{S}_I \ldots]]$$

$$= \sum_{S_i} \exp[-H[K_1 \ldots; S_1 \ldots]]$$

$$\rightarrow Z(\vec{K}; N) = Z(\vec{K}'; \frac{N}{b^D})$$

free energy: $g(\vec{K}) = \lim_{N\to\infty} \frac{1}{N} \ln Z(\vec{K}; N)$

$$= \lim_{N\to\infty} \frac{1}{N} \ln Z(\vec{K}'; N/b^d)$$

$$= \frac{g(\vec{K}')}{b^D}$$

→ original Hamiltonian: $\vec{K} = K_1, \ldots$

→ blocked Hamiltonian: $\vec{K}' = K'_1, \ldots$

→ consider it a transformation

→ consider it a transformation $\vec{K} \to \vec{K}'$

$\Rightarrow \vec{K}' = \vec{T}(\vec{K})$

$\vec{T}(\vec{K}) \to$ vector function

→ repeated renormalization steps generate a trajectory in parameter space

$$\vec{K} \xrightarrow{b^D} \vec{K}' \xrightarrow{b^D} \vec{K}'' \longrightarrow \vec{K}''' \to \dots$$

→ can guess how the trajectory behaves

* if system is $T > T_c$

finite correlation length ξ

each renormalization decreases the correlation length

$$\xi \to \xi/b \to \xi/b^2 \to \xi/b^3 \to \dots$$

$\Downarrow$

$\xi \to 0 \Rightarrow$ system behaves like the high-temperature system (where all spins are uncorrelated)

$\Downarrow$

high-temperature fixed point

* if the system is $T<T_c$
  - renormalization leads to a completely ordered $(T=0)$ state
  - $\rightarrow$ to see this consider that below T_c the system is ordered, which means that the majority of spins point in one given direction (let's say up)

$\Downarrow$ $\Downarrow$

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each renormalization will have the effect of decreasing the number (proportion) of down-spins $\rightarrow$ eventually it becomes all up spins

$\Downarrow$ $\Downarrow$

low-temperature fixed point

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we found <u>two</u> so called <u>fixed points</u>

at a <u>fixed point</u> it holds that

$$\vec{K} \longrightarrow \vec{K} \quad (\vec{K} = \vec{T}(\vec{K}))$$

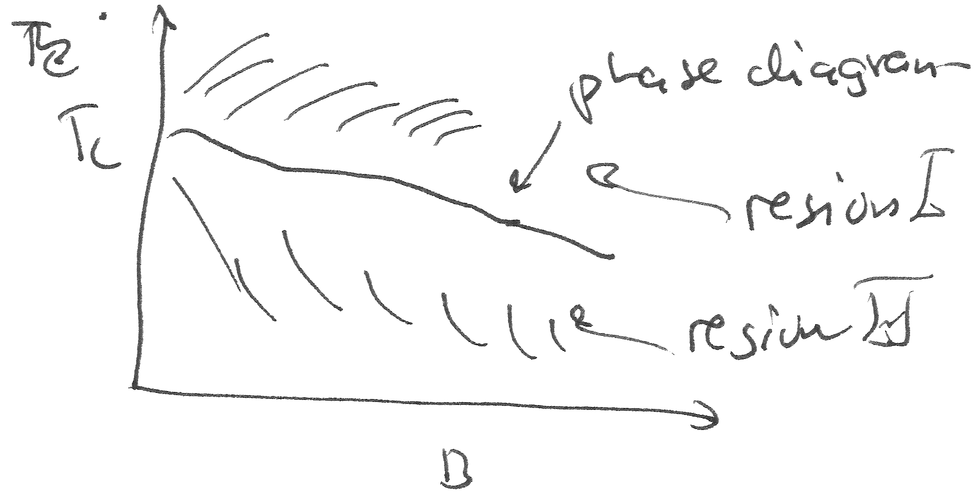
transformation will take the parameters into themselves at fixed points

→ $T < T_c \Rightarrow$ renormalization, applied many times in succession will lead to low-temperature fixed pt.

→ $T > T_c \Rightarrow$ will lead to high temperature fixed pt.

what about T_c itself?

→ keep in mind T_c is a function of other thermodynamic variables → B, etc.



In region I: starting a renormalization trajectory leads to high-T fixed pt.

in region II: starting a renormalization trajectory leads to low-T fixed pt.

on the surface defined by the phase boundary several things can happen

* at T_c we can have

  * $\boxed{\vec{K}^* = \vec{T}(\vec{K}^*)}$ ⇒ <u>critical fixed point</u>

  fixed point on the critical surface
* can also have a line of critical fixed points
* can also have oscillations between a number of pts., strange attractors, etc.

⇓ ⇓ ⇓

we will only consider the first option

→ <u>critical fixed point: $\vec{K}^* = \vec{T}(\vec{K}^*)$</u>

let us assume that the Hamiltonian has two parameters K_1, K_2

$$K_1^* = T_1(K_1^*, K_2^*)$$
$$K_2^* = T_2(K_1^*, K_2^*)$$

let us try to understand how the trajectory behaves <u>near</u> the critical fixed point

start with $\vec{K}$ → close to $\vec{K}^*$

expect that $\vec{K}'$ $(\vec{K} \to \vec{K}')$ will also be close to $\vec{K}^*$

$$\vec{k}' = \vec{T}(\vec{k})$$

expand around $\vec{k}^*$

$$\vec{k}' = \vec{k}^* + \bar{A}(\vec{k} - \vec{k}^*)$$

$$\vec{k}' - \vec{k}^* = \bar{A}(\vec{k} - \vec{k}^*)$$

$$\boxed{\delta\vec{k}' = \bar{A}\,\delta\vec{k}}$$

$\bar{A}$ is the matrix $\quad A_{ij} = \frac{\partial T_i}{\partial k_j}(k_1^*, k_2^*)$

diagonalize matrix leads to

$$\delta\vec{k}' = \bar{A}\,\delta\vec{k} \longrightarrow \begin{pmatrix} \delta u_1' \\ \delta u_2' \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \delta u_1 \\ \delta u_2 \end{pmatrix}$$

$$\delta u_1' = \lambda_1 \delta u_1$$

$$\delta u_2' = \lambda_2 \delta u_2$$

eigenvalues determine behaviour of the trajectory

after n-steps we will have

$$\delta u_1^{(n)} = \lambda_1^n \delta u_1$$

$$\delta u_2^{(n)} = \lambda_2^n \delta u_2$$

if $\lambda_i < 1$ → moves toward fixed point

if $\lambda_i > 1$ → moves away from fixed point

→ relevant eigenvalue

→ measures distance from critical pt.

scaling relation associated with renormalization (assuming $\lambda_1, \lambda_2 \to$ relevant)

$$g(\varepsilon, B) = b^{-D} g(\lambda_1 \varepsilon; \lambda_2 B)$$

looks very much like Widom scaling

$$g(\varepsilon, B) = \lambda^{-1} g(\lambda^p \varepsilon, \lambda^q B)$$

it follows that:

$$\lambda = b^D \qquad \lambda_1 = \lambda^p$$

$$\Rightarrow \lambda_1 = b^{pD}, \quad \lambda_2 = b^{qD}$$

$$\ln \lambda_1 = pD \ln b, \quad \ln \lambda_2 = qD \ln b$$

$$\Rightarrow \boxed{p = \frac{\ln \lambda_1}{D \ln b} \qquad q = \frac{\ln \lambda_2}{D \ln b}}$$

$\Rightarrow$ from determining $\lambda_1, \lambda_2 \Rightarrow$ we can determine p, q $\Rightarrow$ also the critical exponents