Theory of Homogeneous Functions

given a function $f(x)$:

it is homogeneous if $f(\lambda x) = g(\lambda) f(x)$

consequences: $f(\lambda \mu x) = g(\lambda\mu)\, f(x)$

$$= g(\lambda)\, f(\mu x)$$

$$= g(\lambda) g(\mu)\, f(x)$$

$\Rightarrow \; g(\lambda\mu) = g(\lambda) g(\mu)$

$\xrightarrow{\frac{d}{d\mu}} \quad \lambda g'(\lambda\mu) = g(\lambda)\, g'(\mu)$

set $\mu = 1 \Rightarrow \lambda g'(\lambda) = g(\lambda)\; g'(1)$

$\Rightarrow \; g(\lambda) = \lambda^p$

$g'(\lambda) = p\lambda^{p-1}$

$g'(1) = p$

$\underbrace{\lambda g'(\lambda)}_{\lambda [p \lambda^{p-1}]} = \lambda^p\, p$ ✓

- can generalize to higher dimensions: $F(x,y)$

homogeneous if

$$\boxed{F(\lambda^p x, \lambda^q y) = \lambda F(x,y)}$$

Scaling Theory (Widom)

free energy: $f(t, B) \Rightarrow t = \frac{T - T_c}{T_c} \Rightarrow$ temperature

B = magnetic field

assume $f(t, B)$ is a homogeneous fu. of t, B

$$f(\lambda^p t, \lambda^q B) = \lambda f(t, B)$$

$\Rightarrow$ it is possible, based on this assumption, to derive relations between critical exponents (scaling relations)

$\Rightarrow$ there are two scaling forms for the free energy:

1) set $\lambda^p t = 1 \Rightarrow \lambda = t^{-1/p}$

$$f(t, B) = t^{1/p} f(1, t^{-q/p} B)$$

2.) set $\lambda^q B = 1 \quad \lambda = B^{-1/q}$

$$f(t, B) = B^{1/q} f(B^{-p/q} t, 1)$$

- scaling relations are obtained by expressing the thermodynamic quantities from the free energy and ~~to~~ comparing to the definitions of critical exponents let $t=0 \Rightarrow f(t,B) = B^{1/q} f(0,1)$

$$M = \frac{\partial f}{\partial B} = \frac{1}{q} B^{\frac{1}{q}-1} f(0,1)$$

definition of δ: $M \sim B^{1/\delta} \Rightarrow \boxed{\delta = \frac{q}{1-q}}$

---

let $B=0 \Rightarrow f(t,0) = t^{1/p} f(1,0)$

$$E \sim \frac{\partial f}{\partial t} = \frac{1}{p} t^{\frac{1}{p}-1} f(1,0)$$

$$C \sim \frac{\partial^2 f}{\partial t^2} = \frac{1}{p}\left(\frac{1}{p}-1\right) t^{\frac{1}{p}-2} f(1,0)$$

definition of α: $C \sim t^{-\alpha} \Rightarrow \boxed{\alpha = \frac{2p-1}{p}}$

---

start with $f(t,B) = t^{1/p} f(1, t^{-q/p} B)$

$$M \sim \frac{\partial f}{\partial B} = t^{\frac{1-q}{p}} f(1, t^{-q/p} B)$$

set $B=0$

$$M \sim t^{\frac{1-q}{p}} f(1,0)$$

definition of β: $M \sim t^{\beta} \Rightarrow \boxed{\beta = \frac{1-q}{p}}$

---

continue previous:

$$\chi \sim \frac{\partial M}{\partial B} = t^{\frac{1-2q}{p}} f(1, t^{-q/p} B)$$

$B=0$

$$\chi \sim t^{\frac{1-2q}{p}}$$

definition of γ: $\chi \sim t^{-\gamma}$ $\boxed{\gamma = \frac{2q-1}{p}}$

these four relations lead to scaling relations

$$\gamma = \left(\frac{2q-1}{p}\right)$$

$$\beta(\delta-1) = \left(\frac{1-q}{p}\right)\left(\frac{q}{1-q} - \left(\frac{1-q}{1-q}\right)\right) = \left(\frac{1-q}{p}\right)\left(\frac{2q-1}{\cancel{1-q}}\right)$$

$$= \left(\frac{2q-1}{p}\right)$$

$$\boxed{\gamma = \beta(\delta-1)}$$

$$\alpha = \frac{2p-1}{p}$$

$$\beta\delta = \left(\frac{1-q}{p}\right)\left(\frac{q}{1-q}\right)$$

$$\beta = \frac{1-q}{p}$$

$$\alpha + \beta\delta + \beta = \frac{2p-1}{p} + \frac{q}{p} + \frac{1-q}{p} = 2$$

$$\boxed{\alpha + \beta(\delta+1) = 2}$$

---

express p, q:

$$\boxed{p = \frac{1}{\beta(\delta+1)}}$$

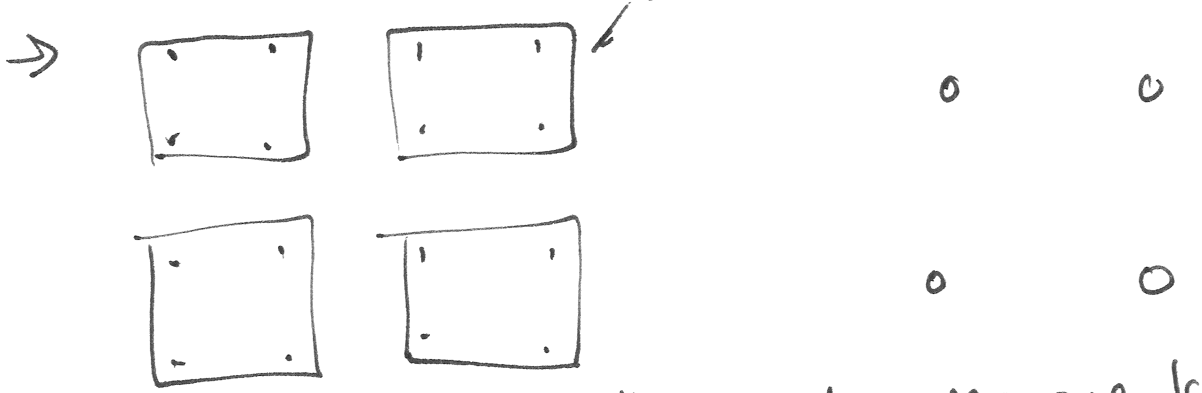
$$\boxed{q = \frac{\delta}{\delta+1}}$$

(useful later)

Kadanoff scaling

- compliments Widom scaling in the sense that it derives scaling relations for η, ν
- based on linear scaling of the spin variables
- main concept: <u>blocking</u> → also the basis of renormalization theory
- idea of blocking: given a lattice ⇒ ~~square~~ form blocks of lattice sites in such a way that the blocks themselves form the same lattice as the original one

lattice   blocks   blocked lattice



from square lattice   to   square lattice

Hamiltonian: (square lattice)

$$H[S] = -K \sum_{ij} S_i S_j - B \sum_i S_i$$

form blocks of b^D ⇒ b^D sites form a block

assume $b \ll \xi$ ξ - correlation length

- with each block associate a spin variable

  1.) define $S'_I = \sum_{i \in I} S_i$

  I - block index

  2.) $S'_I = Z S_I$, with Z ↑ scaling parameter

- assume that $Z = b^y$ (justification: two successive blockings ~~b'~~ b, b' can be thought of as one blocking b'' $\Rightarrow b'' = bb'$)

- can form a new Hamiltonian based on ~~S_I~~ blocked spins:

$$H[S_I] = -K_b \sum_{IJ} S_I S_J - B_b \sum_I S_I$$

⇓

this Hamiltonian will have new parameters t_b and B_b

free energy: $\underbrace{\boxed{f(t_b, B_b) = b^d f(t, B)}}_{\text{similar to scaling form}}$

also expect that upon blocking the correlation length scales with b the size of the block $\Rightarrow$ $\boxed{\xi' = \frac{\xi}{b}}$

B_L can be easily determined:

$$B_b \sum_I S_I \to \frac{B_b}{Z} \sum_I S_I' = \frac{B_b}{Z}\sum_I \sum_{i\in I} S_i = \underbrace{\frac{B_b}{Z}\sum_i S_i}_{B\sum_i S_i}$$

$$\boxed{B_b = Z B}$$

$$\boxed{B_b = b^y B}$$

assume that $t_L = t b^x$

$$\Rightarrow \quad \boxed{f(b^x t, b^y B) = b^D f(t, B)}$$ $\Leftrightarrow$ scaling relation for a homogeneous function

$$\Rightarrow \quad \boxed{x = pd \,, \quad y = qD}$$

<u>correlation function:</u>

$$C(r_b; t_b) = \langle S_I S_J\rangle - \langle S_I\rangle\langle S_J\rangle$$

$$= \cancel{\sum_{i\in I}\sum_{j\in J}} = \frac{1}{Z^2}\left[\langle S_I' S_J'\rangle - \langle S_I'\rangle\langle S_J'\rangle\right]$$

$$= \sum_{i\in I}\sum_{j\in J}\frac{1}{Z^2}\left[\langle S_i S_j\rangle - \langle S_i\rangle\langle S_j\rangle\right]$$

if $b \ll \xi$ $\Rightarrow$ expect that

$$\langle S_i S_j\rangle - \langle S_i\rangle\langle S_j\rangle \approx \langle S_{i'} S_{j'}\rangle - \langle S_{i'}\rangle\langle S_{j'}\rangle$$

$$i, i' \in I \qquad j, j' \in J$$

$$\Downarrow$$

$$C(r_b, t_b) = b^{2D-2y}\, C(r, t)$$

$$= b^{2(D-y)}\, C(r, t)$$

since $r \to$ distance $\Rightarrow$ it scales as $r_b = r/b$ (8)

$$C(\tfrac{r}{b}, t_b) = b^{2(D-y)} C(r,t)$$

$$C(\tfrac{r}{b}, b^x t) = b^{2(D-y)} C(r,t)$$

set $\frac{r}{b} = 1$ consider $t=0$ (at T_c)

$$C(1,0) = r^{2D-2y} C(r,0)$$

$$\Rightarrow \boxed{C(r,0) \sim \frac{1}{r^{2D-2y}}}$$

we know that at T_c $C(r) \sim \frac{1}{r^{D+\eta-2}}$

$$\Rightarrow \quad D+\eta-2 = 2D-2y$$

$$2y = D-\eta+2$$

$$\boxed{y = \frac{D-\eta+2}{2}}$$

away from T_c we have

$$C(r,t) = \frac{C(\frac{r}{b}, b^x t)}{b^{2(D-y)}}$$

set $r/b = 1 \Rightarrow r = b$

$$C(r,t) = \frac{C(1, r^x t)}{r^{2(D-y)}}$$

since $C(r,t) = \dfrac{f(r/\xi)}{r^{D-2+\eta}} \Rightarrow r\xi^{-1} = rt^{1/x}$

$$\xi \sim t^{-1/x}$$

since $\xi \sim t^{-\nu} \Rightarrow \boxed{\nu = 1/x}$

- after some algebra one can derive the scaling relations:

$$\boxed{\nu = \frac{2-\alpha}{d}}$$

$$\boxed{\eta = 2 - \frac{d\gamma}{2\beta+\gamma}}$$